Duality between d and $\mathcal{L}$ (V)

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Summary

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Quick Recap

- Let's summarize what we've seen so far.
- A LA L is dual to a LCA L^*.
- The Jacobi identity is dual to the cojacobi identity, which we wrote in two equivalent ways (original and "bridge" form).
- The cojacobi identity is equiv to the usual $d^2 = 0$ cocycle condition, with d the unique extension of Δ on Λ^1 to an anti-derivation on Λ, given some choice of action of d on the ground field/ring Λ^0.
- The Jacobi identity and cojacobi identity (and thus cocycle condition) all contain the same information, encapsulated in structure constants c^i_{jk} for the LA and $-c^i_{jk}$ for the LCA.
- I.e., the cojacobi identity is the cocycle condition, but only on the extension, not the LCA itself.
- Recall from DG:
  - $[,]$ for v.f.s is a LA on $\Gamma(T^1_0(M))$.
  - $\mathcal{L}$ is the unique deriv ext of this $[,]$ to all of $T(M)$ (and $\Lambda(M)$) with action on $T^0 = \Lambda^0$ given by $\mathcal{L}_V(f) = V(f)$
  - d is defined directly on $\Lambda(M)$ itself, and is not grade-preserving.
- We'll now see the connection between them, but before we do so it is necessary to address a crucial point we alluded to earlier.

The choice of ground field or ring (I)

- In moving from our discussion of LA/LCA's in general to the formulations of $\mathcal{L}$ and d used in DG, we must address a key point.
- We mentioned that there are two equiv. ways of viewing $T(M)$ and $\Lambda(M)$ as graded algebras.
  - As ∞-dim v.s.'s over K (i.e. $\mathbb{R}$ or $\mathbb{C}$)
  - As finite-basis modules over $C^\infty(M)$.
- Though mathematically interchangeable, the two views have quite different conceptual properties.
  - The v.s. view is useful when discussing LA's, LCA's, and general properties.
  - The module view is what we encounter in DG, and work with.
- So far, our discussion has revolved around the v.s. view. Now we move to the module view.
- The conseqs are nontrivial, and explain something we glossed over: why do we care about the extension being a deriv/anti-deriv? We've taken this as a given, since we're so used to dealing with such things from DG — and know that we're aiming for a duality between d and $\mathcal{L}$ — but remember that we started with just a Lie Bracket of v.f.'s. There's no obvious reason why derivs and anti-derivs should arise or be of importance. The transition from K to $C^\infty(M)$ answers this.

The choice of ground field or ring (II)

- Let's first consider a simple example of the difference in DG: the v.f.s on M (i.e. $\Gamma(TM)$).
- $\Gamma(TM)$ is an ∞-dim v.s. over K, with a natural LA structure.
- The (Hamel) basis is some infinite set of v.f.'s on M. I.e., any v.f. can be written as a finite linear combo of these.
- In some cases, there may be a Schauder basis as well (ex. square-integrable, etc) but we won't worry about that here.
- Note that the basis is not a point-by-point direct sum of T_pM bases. This wouldn't account for smoothness or admit a LA, both of which draw on information beyond the single point p.
- Note also that a basis is *not* a section of LM (trivializing that PB), because we are working over K, not $C^\infty(M)$ here!
- The LA commutator is defined via $[v, w](f) \equiv v(w(f)) - w(v(f))$. It is linear, so its action c^i_{jk} on the basis determines its action on all v.f.'s. However, c^i_{jk} has infinitely many elements.
- $[v, w]$ is antisymmetric and bilinear, as needed for a LA. I.e. $[v, aw + z] = a[v, w] + [v, z]$ for $a \in K$.
- The LA thus defined on $\Gamma(TM)$ has a LCA, but it is on $\Gamma^*(TM)$, not $\Gamma(T^*M)$. The relevant map is Δ, as defined earlier.

The choice of ground field or ring (III)

- What if we view it over $C^\infty(M)$ instead?
- Now, $\Gamma(TM)$ has finitely many basis elements. If globally defined, these would correspond to a section of LM — which does not exist unless LM is trivial. However, for our purposes we don't require a global def. $[,]$ only draws on local info, and there need only be a local section. This always exists.
- If $n = \dim M$, any v.f. can be expanded locally as $\sum_{i=1}^{n} f_i e^i$, with e^i the basis v.f.'s and f_i smooth fns on M.
- The problem is that $[,]$ no longer is linear.
- $[v, fw + z] \neq f[v, w] + [v, z]$. It still distributes over addition, but scalar mult no longer passes through!
- Modules have a notion of linearity, just as v.s.'s do. However, a fn linear in K on V may not be linear in $C^\infty(M)$.
- Put another way, linearity is tied to the view of $T(M)$ we take.
- There is a LA on $T_0^1(M)$ over K, but not over $C^\infty(M)$.
- We still have a Lie Bracket, but now it has a different behavior. It is not linear. From DG, $[v, fw] = \mathcal{L}_v(fw) = f\mathcal{L}_v(w) + \mathcal{L}_v(f)w$.
- This may be rewritten $[v, fw] = f[v, w] + \mathcal{L}_v(f)w$.
- I.e., $\mathcal{L}_v(f)$ is the obstruction to linearity.

The choice of ground field or ring (IV)

- Similarly, consider d. Defined via the de Rham complex, we have $d(f\omega) = df \wedge d\omega$ rather than $f \wedge d\omega$. It is not linear.
- d is in fact linear over K, but we never work over K (which would involve an infinite Hamel basis for forms).
- Note that when we speak of "over K" vs "over $C^\infty(M)$" we are speaking of v.s. vs module — not simply K as the constant functions in $C^\infty(M)$. Yes, d is linear in K viewed as a subset of $C^\infty(M)$. But that doesn't matter to us.
- What we've seen is that in going from the view over K to that over $C^\infty(M)$, we lose linearity and replace it with a deriv/anti-deriv. This is why we care about derivs/anti-derivs. Bilinear $\rightarrow$ deriv, cobilinear $\rightarrow$ anti-derivation. Because we work with tensor prods, we have linearity in both cases — so this distinction is obscured.
- When we extend $[,]$ to $T(M)$ or Δ to $\Lambda(M)$, we could extend them linearly over K. Equiv to extending as deriv/anti-deriv over $C^\infty(M)$.
- We cheated a little in our earlier discussion. We were working over K, yet invoked uniqueness thms about extensions of $[,]$ and d to derivs/anti-derivs on all of $T(M)$ or $\Lambda(M)$. Strictly speaking, this was inaccurate. The extensions actually were linear over K, and we jumped the gun.

The choice of ground field or ring (V)

- We'll now rephrase, but let's first consider one other thing.
- The extensions we spoke of also relied on a choice of action on the 0-grade. This differs depending on our view.
- The 0-grade in both $T(M)$ and $\Lambda(M)$ is $C^\infty(M)$. Viewed over K, this is an ∞-dim v.s. (just like the v.f.'s are). In that case, the action of d is linear. It maps each basis element to its differential, and is a linear op on the coeffs. Suppose $f = \sum_i c_i g_i$, where the $c_i \in K$, the $g_i \in C^\infty(M)$, and the sum is finite (since Hamel basis). Then $df = \sum_i c_i dg_i$, where each dg_i also is a smooth fn, so we've just performed a linear operation, changing g_i for dg_i.
- However, if we view it over $C^\infty(M)$, then it has a single basis element. Any smooth nowhere-0 fn g will do. Then $f = (f/g)g$, and $df \neq (f/g)dg$. We replace linearity with the Leibnitz rule.
- If we choose an action in one view, its behavior will be determined for the other. I.e., the extensions must be consistent.

The choice of ground field or ring (VI)

- I.e., the linear action we choose for df over K must yield the same result as that defined over $C^\infty(M)$.
- Either will do, but we typically think in terms of fns in DG, so defining df over $C^\infty(M)$ tends to be more intuitive. This is $df(v) = v(f)$ (which in local coords is $\sum v^i \partial_i f$).
- Our discussion of LA/LCA duality was over K, so d acted linearly. When working with $\Lambda(M)$ as an alg over $C^\infty(M)$ (as we usually do in DG), the Leibnitz terms arise as well. However, the result must be consistent — and everything (including the cocycle condition) falls out either way. In that particular case, it just was easier to see over K. In DG it arises more naturally over $C^\infty(M)$.
- Similarly, our LA $[,]$ of v.f.'s (viewed over K) has a unique linear ext to $T(M)$, given a choice of behavior on $T^0_0(M)$. This corresponds to a unique deriv ext to $T(M)$ of the same bracket — but no longer linear — over $C^\infty(M)$ to $T(M)$.
- That extension is $\mathcal{L}$ when the action on T^0_0 is that from DG: $\mathcal{L}_v(f) = v(f)$.

The choice of ground field or ring (VII)

- There is an apparent oddity here:
  - The linear ext over $K \leftrightarrow$ the deriv ext over $C^\infty(M)$.
  - The deriv ext is unique up to an action on the 0-grade.
  - The linear ext is unique, period.
  - There seems to be more information in one than the other.
- We also may wonder where the info for df or $\mathcal{L}_V(f)$ comes from, if we just start with $[,]$. After all, df or $\mathcal{L}_V(f)$ usually is where we connect to coord derivs — even in the de Rham formulation.
- It turns out the information content is the same in the two exts. The point of contact to calculus already is present in $[v, w]$, which requires meaning for $v(f)$ through its DG def as a tangent vector (i.e. via coord derivs).
- We have a unique linear ext outward from T^1_0 to everything. Were we to extend as derivs over $C^\infty(M)$ instead, we'd also require an action on T^0_0 for the uniqueness thms because the coeffs would be smooth fns. We'd need to know how to operate on these when they appear in the Leibniz rule. In that case, the $[,]$ alone is insufficient. The Leibniz rule (in lieu of linearity) implicitly assumes knowledge of the action on smooth fns!

The choice of ground field or ring (VIII)

- In our case, $\mathcal{L}_V(f) = V(f)$ and $df(V) = V(f)$ are the correct 0-grade exts over $C^\infty(M)$.
- Put another way, when we glibly said that linearity $\rightarrow$ Leibnitz, we really meant that linearity over K on $T^1_0(M)$ corresponds to the Leibnitz rule for both $C^\infty(M)$ and $T^1_0(M)$. The corresponding info resides across two grades instead of one in that case.
- Note that we also require the deriv/anti-deriv ext to commute with contractions and have the grade-adding property in order for the ext to be unique. Those are fine. These are possessed by the deriv/anti-deriv counterpart of the linear extension.
- This is why there was no differential term earlier, but there is in DG. We were constructing a linear extension of the LA in the former case, and the corresponding deriv ext in the latter.
- This also is why we care about derivs/anti-derivs. They are the proxy for linearity when we move to working over $C^\infty(M)$, as we must in DG. They also could be viewed as surrogates for it or approxs to it.

The choice of ground field or ring (IX)

- To summarize:
  - Linearity over K becomes deriv/anti-deriv over $C^\infty(M)$.
  - $\mathcal{L}_V(f)$ is the obstruction to linearity for $[,]$.
  - df is the obstruction to linearity for Δ.
- Let us also summarize some similarities/diffs between d and $\mathcal{L}$.
  - $\mathcal{L}$ extends a LA and d extends a LCA
  - $\mathcal{L}$ is on $T(M)$ or $\Lambda(M)$, and d is on $\Lambda(M)$
  - They have the same action on f
  - $\mathcal{L}$ is a deriv, d is an anti-deriv
  - $\mathcal{L}$ is grade-preserving, d is grade-changing
  - Both can be viewed as obstructions to linearity
  - Or both can be viewed as proxies for linearity
  - Or both can be viewed as approximations to linearity

Duality (I)

- It may look like we have a duality between $[,]$ and d, and between the Jacobi identity and cocycle condition, but this is not quite right.
- We have exact duality between the LA of v.f.'s with $[,]$, and its LCA. However, the dual to $[,]$ is not an operation on the LCA, but rather a map that takes us outside it.
- The duality of the cocycle condition and Jacobi identity involves not the LA and LCA, but rather their extensions.
- Moreover, we typically prefer to work with both $\mathcal{L}$ and d over $C^\infty(M)$ rather than K. I.e., as derivs/anti-derivs.
- We also must decide what space we care about. $\mathcal{L}$ could reside on $T(M)$ or $\Lambda(M)$, but d is on $\Lambda(M)$. Note that $\Lambda(M)$ is not dual in any obvious sense to $T(M)$ as v.s.'s or modules or algebras.
- Recall that $\mathcal{L}$ on $\Lambda(M)$ is *not* merely a restriction to $\Lambda(M)$ as a vector/module subspace. It is compatible with the algebra on $\Lambda(M)$ — i.e. the $\wedge$ product. It remains a deriv in this context [This is evident from Cartan's formula. $\mathcal{L}_X\omega = i_X d\omega + d(i_X\omega)$, both of which are well-defined on wedge products. Each term involves two anti-derivs, which cancel to a deriv.]
- $\mathcal{L}_V(f) = V(f)$ and $df(V) = V(f)$. These are not dual (at least in the same way as the LA/CLA).

Duality (II)

- Given the v.f. commutator on M, we thus have the following:
  - A LA over K on $T_0^1(M)$.
  - A unique linear extension of $[,]$ to $T(M)$ (including $T_0^0(M)$) over K
  - A corresponding unique derivation $\mathcal{L}$ on $T(M)$ over $C^\infty(M)$. It has $\mathcal{L}_V(f) = V(f)$.
  - A unique linear extension of $[,]$ to $\Lambda(M)$ over K, which also has $\mathcal{L}$ as the corresponding unique deriv over $C^\infty(M)$.
  - A LCA over K on $T_1^0(M) = \Lambda^1(M)$, with operation Δ.
  - A unique linear extension of Δ to all of $\Lambda(M)$.
  - A corresponding unique anti-derivation d on $\Lambda(M)$ over $C^\infty(M)$. It has $df(V) = V(f)$, and obeys the cocycle condition.
- $\mathcal{L}$ and d are dual in the sense that they are the unique linear extensions to $T(M)$ and $\Lambda(M)$ of the LA structure $[,]$ on $T_0^1(M)$ and its dual LCA on $\Lambda^1(M)$. We usually see them written in their deriv/antideriv forms over $C^\infty(M)$ instead of their linear forms over K.

Duality (III)

- I.e., we start with a graded tensor algebra. The LA on one particular grade of it uniquely extends to a structure on all of it. This structure is $\mathcal{L}$. The corresponding LCA uniquely extends to a structure on the graded algebra $\Lambda(M)$. This structure is the de Rham complex. The reason the latter is on Λ and not T arises naturally due to the antisymmetry property.
- Both the LA and LCA have antisymm properties, but because Δ moves us between grades, the antisymmetry cannot be divorced from the graded algebra mult. I.e., $[,]$ and $\otimes$ are distinct on $T(M)$, but Δ cannot be distinct from the graded alg mult. It requires antisymm of it, producing $\Lambda(M)$.
- In summary, $\mathcal{L}$ and d derive from dual structures and contain precisely the same info. We thus can speak of them as dual-like.
- What about i_V? Much of what we did with d works with i_V too (with a few mods), and $\mathcal{L}_V$ is symm in its treatment of d and i_V under Cartan's formula. However, i_V is crucially different from d and does not arise from the LCA. Unlike d, i_V depends on V and is at best dual to $\mathcal{L}_V$, not $\mathcal{L}$. In some sense, we can think of i_V as the currying mechanism which converts us between $\mathcal{L}$ and $\mathcal{L}_V$ in the eyes of d.

Some references (I)

- Tensor Algebras, Exterior Algebras, Anti-derivations, and most of the uniqueness results we discussed: Kobiyashi & Nomizu has an excellent development in the 1st chapter of book 1.
- Def of de Rham complex: Bott & Tu 1st few pages.
- Dual of chain complex: https://math.stackexchange.com/questions/1032596/how-does-one-actually-take-the-dual-of-a-chain-complex
- Infinite dimensional LG of a LA: https://math.stackexchange.com/questions/1285996/is-there-an-infinite-dimensional-lie-group-associated-to-the
- Discusses unique extension of D to anti-derivation on Λ https://math.stackexchange.com/questions/369438/anti-derivation-of-an-exterior-algebra
- Covers Lie Coalgebra, def of ops. https://en.wikipedia.org/wiki/Lie_coalgebra
- How to define a Lie Coalgebra (with concrete example for sl2): https://math.stackexchange.com/questions/167472/definition-of-the-lie-coalgebra
- Discussion of bilinearity vs linearity ($\otimes$ vs $\times$): https://www.math.ucla.edu/~mikehill/Teaching/Math5651/Lecture16.pdf

Some references (II)

- Excellent discussion of structure constant relationship (using a basis): https://math.stackexchange.com/questions/1706438/relation-between-exterior-derivative-and-lie-bracket
- Dual Jacobi identity (but for Lie Bialgebra): https://math.stackexchange.com/questions/2406942/dual-jacobi-identity-for-lie-bialgebra
- Lie Algebra/Lie coalgebra: https://math.stackexchange.com/questions/1652169/relations-between-lie-algebras-and-lie-coalgebras?rq=1
- Review of original Lie Coalgebra formulation (Walter Michaelis, Advances in Mathematics 38, p. 1-54 (1980)): https://core.ac.uk/download/pdf/82169049.pdf
- Cartan's Formula (the treatment we follow): https://unapologetic.wordpress.com/2011/07/26/cartans-formula/
- Also interesting: https://unapologetic.wordpress.com/2011/07/28/the-lie-derivative-on-cohomology/