Duality between d and $\mathcal{L}$ (IV)

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Summary

- Cartan's Formula
- Review of tensor product vs direct product
- Element free LA Def
- Lie Coalgebras

Cartan's Formula (I)

- To motivate, let f be a smooth fn and X, Y be v.f.s:
  - Fns: $\mathcal{L}_X f = X(f)$ and $i_X(df) = df(X) = X(f)$. I.e. $\mathcal{L}_X = i_X \circ d$ when applied to fns.
  - Fns: We can write $d \circ i_X = 0$ for fns. Technically i_X is not defined on a 0-form (i.e. fn), but for some purposes (such as this), it makes sense to declare it to be 0.
  - Exact 1-form : $i_X \circ d(df) = i_X \circ d^2 f = 0$.
  - Exact 1-form : $(\mathcal{L}_X(df))(Y) = Y(X(f))$. On the other hand, $(d \circ i_X)(df) = d([df(X)])$. The right side is not 0 because d operates on $df(X)$, not df. $df(X) = X(f)$, so we have $d(X(f))$. $X(f)$ is a fn, so this is a 1-form, and $d(X(f))(Y) = Y(X(f))$. I.e., $\mathcal{L}_X = d \circ i_X$ when applied to exact 1-forms.
- We thus have $\mathcal{L}_X = d \circ i_X + i_X \circ d$ when applied to both fns and exact 1-forms (though different terms are 0 in the 2 cases!).
- Cartan's formula: $\mathcal{L}_X = d \circ i_X + i_X \circ d$ for all forms.
- Note that $\mathcal{L}_X$ does not change the degree of the form. d raises the degree and i_X lowers the degree, and in both terms we are back where we started.

Cartan's Formula (II)

- $\mathcal{L}_X$ maps the de Rham complex to itself, maintaining the form degrees. I.e., it maps $\Lambda^k \to \Lambda^k$.
- $\mathcal{L}_X \circ d = d \circ \mathcal{L}_X = d \circ i_X \circ d$. I.e., $\mathcal{L}_x$ commutes with d.
- This means $\mathcal{L}_X$ is a chain map from the de Rham complex to itself.
- As such, it induces autohomomorphisms of the cohomology groups H^k of the de Rham complex.
- To compute them, we must consider $\mathcal{L}_X\omega$ for a closed form ω (since the cohomology is just the closed forms over the exact forms).
- For closed forms, $d\omega = 0$ and Cartan's formula reduces to $\mathcal{L}_X\omega = d \circ i_X\omega$. This can be written $d(i_X\omega)$, making clear that $\mathcal{L}_X\omega$ is an exact form.
- $\mathcal{L}_X$ maps any closed form to an exact form, nullifying the cohomology by sending any element of H^k to the identity of H^k. Such a chain map has all of H^k as its kernel, and is called null-homotopic.

Review of tensor product vs direct product (I)

- Let's quickly review the difference between a tensor product $V \otimes W$, direct product $V \times W$, and direct sum $V \oplus W$ of linear spaces, something we will use shortly when developing Lie coalgebras.
- Let $\{e_i\}$ and $\{f_i\}$ be bases for V and W.
  - Direct product (aka cartesian product) $V \times W$. All pairs (v, w). It is a linear space with basis the disjoint union $\{e_i\} \sqcup \{f_i\}$. Any element can be written $\sum v^i e_i + \sum w^j f_j$.
  - Direct sum $V \oplus W$. Same as direct prod for finite products, but not infinite prods. Direct Prod is all seqs, and direct sum is all seqs with only a finite number of nonzero elements. We'll work with finite prods, so the two are the same.
  - Tensor product $V \otimes W$. Has basis $\{e_i\} \times \{f_j\}$. Any element can be written $\sum z^{ij} e_i \otimes f_j$.
- Both $V \times W$ and $V \otimes W$ are linear spaces.
- Tensor products convert the notion of multilinearity to a notion of linearity. Let's see what this means.
- Often, we encounter maps of the form $f : V \times W \to U$ which are bilinear, meaning linear in each argument separately.
- Ex. the internal mult of an algebra $\cdot : V \times V \to V$.

Review of tensor product vs direct product (II)

- Bilinearity means $f(av+w,z) = af(v,z) + f(w,z)$ and $f(v, az+w) = af(v,z) + f(v,w)$. We also may have sesquilinear or other variants, but the idea is the same. It is linear in each component separately.
- This is different from f being a linear map.
- $V \times W$ is a linear space, inheriting linearity from V and W.
  - Expand $v = \sum v^i e_i + \sum w^j f_j$ and $v' = \sum v'^i e_i + \sum w'^j f_j$.
  - Then $v + av' = \sum (v^i + av'^i) e_i + \sum (w^j + aw'^j) f_j$.
  - Put another way, $(v,w) + a(v',w') = (av + v', aw + w')$.
- $V \otimes W$ also is a linear space, inheriting linearity as well.
  - Expand $z = \sum z^{ij} e_i \otimes f_j$ and $z' = \sum z'^{ij} e_i \otimes f_j$
  - $z + az' = \sum (z^{ij} + az'^{ij}) e_i \otimes f_j$
  - Not all $z \in V \otimes W$ can be expressed as $z^{ij} = v^i w^j$ (i.e. $z = v \otimes w$). If we're dealing with v.s.'s, and $\dim V = \dim W = 10$, then $\dim(V \otimes W) = 100$, while the relevant subspace is isomorphic to $V \times W$, and has $\dim(V \times W) = 20$. Most z's are not of the form $v \times w$, but rather a linear combo of such terms.
  - If we do have $z = v \times w$ and $z' = v' \times w'$ then linearity gives us $(v \times w) + a(v' \times w') = \sum (v^i w^j + a(v'^i w'^j)) e_i \otimes f_j$, which in general does not factor into any $v'' \times w''$.

Review of tensor product vs direct product (III)

- Now consider a bilinear map $f : V \times W \to U$.
  - $f(a(v,w) + (v',w')) = f(av + v', aw + w') =$
    $a^2 f(v,w) + a[f(v,w') + f(v',w)] + f(v',w')$
  - Were f linear, we would have $af(v,w) + f(v',w')$.
  - A bilinear map is not linear.
- When thinking in terms of binary operations in algebra, it is natural to work with bilinear maps and direct products. However, for element-free category theory, linearity is preferable.
- Why? The dual to bilinearity is unintuitive and messy. The dual to linearity is linearity.
- To move from a bilinear map to a linear one involves moving from $V \times W$ to $V \otimes W$.
- Thm: The bilinear maps $V \times W \to U$ are the same as the linear maps $V \otimes W \to U$.
- Although $V \otimes W$ is much bigger than $V \times W$, and it may seem as if there should be many more linear maps to U from it, this is not the case. This is because the information for either is encoded in a linear action on the two bases.
- Both a bilinear map $V \times W \to U$ and a linear map $V \otimes W \to U$ can be described via a $(1,2)$ tensor f^i_{jk}.

Review of tensor product vs direct product (IV)

- Any bilinear map $\eta_\times : V \times W \to U$ induces a unique linear map $\eta_\otimes : V \otimes W \to U$ defined on the basis as $\eta_\otimes(e_i \otimes f_j) = \eta_\times(e_1, f_j)$.
- In both cases, we are defining the same action on each pair (e_i, f_j). Bilinearity extends this one way to $V \times W$, and linearity extends it another to $V \otimes W$.
- This extends to multilinear maps. A multilinear map on $V_1 \times \cdots \times V_n$ corresponds to a linear map on $V_1 \otimes \cdots \otimes V_n$.
- If we have a map $\to V \times W$, we can convert this to a map $\to V \otimes W$, by simply including $V \times W \subset V \otimes W$ via $(v, w) \to v \otimes w$. So, if $f(\mathit{stuff}) = (v, w)$, we can replace that with $f(\mathit{stuff}) = v \otimes w$.
- This gives us a prescription to work entirely with linear maps amongst tensor-product spaces.

Review of tensor product vs direct product (V)

- A number of the maps we work with when defining Lie coalgebras can be viewed either way (as a multilinear map on the direct prod or a linear map on the tensor prod). These include the following:
  - $[,]$ for a LA can be defined as a map $\phi : L \times L \to L$ that is bilinear or a map $\phi : L \otimes L \to L$ that is linear.
  - τ is a swap map. It can be defined as $\tau : V \times W \to W \times V$ via $\tau(v, w) = (w, v)$ or as $V \otimes W \to W \otimes V$ via $\tau(e_i \otimes e_j) = e_j \otimes e_i$.
  - ξ is the index shift map. It is defined as $\xi : L \times L \times L \to L \times L \times L$ via $\xi(e_i, e_j, e_k) = (e_j, e_k, e_i)$ or $\xi : L \otimes L \otimes L \to L \otimes L \otimes L$ via $\xi(e_i \otimes e_j \otimes e_k) = e_j \otimes e_k \otimes e_i$.
- Note: there are purer, category-theory ways of defining ξ and τ as well, ones which do not rely on elements or bases.
- Note: the tensor product often is *defined* via the above thm. I.e., it is the unique linear space s.t. the linear maps from it to any U are the bilinear maps from $V \times W$ to U. This is a universal property used to define $V \otimes W$. [See `https://www.math.ucla.edu/~mikehill/Teaching/Math5651/Lecture16.pdf` for details].

Element free LA Def (I)

- Just as we can define a coalgebra of any associative algebra, we can define a Lie coalgebra (LCA) of any LA.
- We'll follow the treatment in "Lie Coalgebras" by Walter Michaelis, Advances in Mathematics 38, p. 1-54 (1980). It is available here: https://core.ac.uk/download/pdf/82169049.pdf. According to Mr. Internet, he is the one who came up with the notion of a Lie coalgebra, at least in the way it is used today.
- A LA is a v.s. which has an antisymmetric mult $[,]$ that obeys the Jacobi identity. The usual formulation of these is element-dependent and not easily dualizable. To dualize, we first cast the def in an element-free form (i.e. make it category theory friendly).
- Standard def of LA: v.s. L with bilinear map $[,] : L \times L \to L$ s.t.
  - $[x, y] = -[y, x]$
  - $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$
- Note that the 1st cond can also be written $[x, x] = 0$ due to linearity.

Element free LA Def (III)

- As mentioned, we can write $[,]$ as the bilinear map $\phi : L \times L \to L$ given by $\phi(x, y) \equiv [x, y]$ or the linear map $\phi : L \otimes L \to L$ given by $\phi(e_i \otimes e_j) = [e_i, e_j]$.
- The first thing we'll do is convert the 2nd requirement. For this, we will use the index shift map ξ.
- Denote by I_L the identity on L and by I_{L^3} the identity on $L \otimes L \otimes L$. We now are in a position to write the Jacobi identity.
- Jacobi identity: $\phi \circ (I_L \otimes \phi) \circ (I_{L^3} + \xi + \xi^2) = 0$.
- As for the anticommutativity condition, I won't go through the machinations, because they are not particularly enlightening (they can be found in Michaelis' paper). For underlying fields of char $\neq 2$, the correct non-element generalization is $\ker\,(1 - \tau) \subset \ker\,\phi$.
- Let's briefly consider what this means and why it makes sense.

Element free LA Def (IV)

- For simplicity, let's assume L is a real, finite-dim vector space.
- Any element of $L \otimes L$ can be written $z = z^{ij}(e_i \otimes e_j)$ in basis $\{e_i\}$.
- ϕ is linear, so write $\phi(e_i \otimes e_j) \equiv \phi_{ij}^k e_k$ in this basis.
- $\phi \equiv [,]$, so the element-wise antisymm cond is $\phi_{ij}^k = -\phi_{ji}^k$.
- $\phi(z) = z^{ij}\phi(e_i \otimes e_j) = z^{ij}\phi_{ij}^k e_k$. $z \in \ker\, \phi$ iff $z^{ij}\phi_{ij}^k = 0$ for each k.
- $\tau(z) = z^{ij}(e_j \otimes e_i) = z^{ji}(e_i \otimes e_j)$. I.e., τ swaps z_{ij}'s indices.
- $(1-\tau)(z) = (z^{ij} - z^{ji})e_i \otimes e_j$. $z \in \ker\,(1-\tau)$ iff z_{ij} is symm.
- $\ker\,(1-\tau) \subset \ker\, \phi$ says that all symm z^{ij} nullify $z^{ij}\phi_{ij}^k$.
- Although these aren't matrices (i.e. $(1,1)$ tensors), we can pretend without issue because we have 2 upper and 2 lower indices. The product and trace rules for matrices hold, and are basis indep in this particular context. For antisymm matrix M, $tr(SM) = 0$ for all symm matrices S.
- The converse holds too (if $tr(SM) = 0$ for all symm matrices, then M is antisymm), but only for char $K \neq 2$.

Element free LA Def (V)

- $z \in \ker(1-\tau)$ thus requires ϕ_{ij}^k antisymm in i, j (for each k). Each such symm z then would be in $\ker\phi$ as well.
- Another way of putting this is that $\mathrm{Im}\,(1+\tau) \subset \ker(1-\tau)$. I.e., the symmetric matrices antisymmetrize to 0.
- Equality holds when char $\neq 2$, consistent with our earlier remark that the constraint on the exterior alg is stronger than mere antisymmetrization when char=2.
- So, for our purposes, the constraint $[x, y] = -[y, x]$ is equivalent to requiring that all elements which nullify $(1-\tau)$ also nullify ϕ.
- For the char$\neq 2$ case, we could just say $\ker(1-\tau) = \ker\phi$, but in general we must write $\ker(1-\tau) \subset \ker\phi$.
- We also may write the antisymm condition (for char $\neq 2$) as $\phi = -\phi \circ \tau$.
- It is easy to see that these results are basis-indep.

Lie Coalgebra (I)

- To summarize, a LA is a v.s. with a map $\phi : L \otimes L \to L$ s.t.
  - ker $(1-\tau) \subset$ ker ϕ
  - $\phi \circ (I_L \otimes \phi) \circ (I_{L^3} + \xi + \xi^2) = 0$
- Dualization maps L to its dual v.s., reverses all arrows, flips the order of $\subset$, and swaps ker $\leftrightarrow$ Im .
- A Lie Coalgebra (LCA) is a v.s. X and a map $\Delta : X \to X \otimes X$ s.t.
  - Im $\Delta \subset$ Im $(1-\tau)$
  - $(I_{X^3} + \xi + \xi^2) \circ (I_X \otimes \Delta) \circ \Delta = 0$
- We also could express the 1st condition (antisymmetry) as $\phi = -\phi \circ \tau$, and its dual as $\Delta = -\tau \circ \Delta$.
- A quick reminder that this is where expressing things as linear maps (and thus using the tensor product) proves its worth. Had we expressed things in terms of direct products and multilinear maps, dualizing would be nontrivial.

Lie Coalgebra (II)

- Pick basis $\{e^i\}$ for X. Defining $\Delta(e^i) = \Delta^i_{jk} e^j \otimes e^k$ in this basis, we see from the 1st cond ($\Delta = -\tau \circ \Delta$) that $\Delta(e^i) = -\tau(\Delta^i_{jk} e^j \otimes e^k) = -\Delta^i_{jk} e^k \otimes e^j = -\Delta^i_{kj} e^j \otimes e^k$. I.e., $\Delta^i_{jk} = -\Delta^i_{kj}$. The 1st condition still represents antisymmetry!
- Note that since $\Delta(x)$ is an element of $X \otimes X$, we already are outside of X. Whereas, ϕ can be viewed as a binary operation on L, Δ necessarily maps us to a different space. We must look beyond X.
- Put another way, unlike $[,]$ on the LA, the LCA map Δ cannot be viewed as an n-ary op on X (for any n). The dual of linear is linear, but the dual of bilinear is not bilinear. It is something unintuitive.
- The obvious extension, since we have antisymmetry, is the exterior algebra. I.e., $\Lambda(L)$, which is built from the dual $X = L^*$.
- We thus see that $\Delta(x) \in \Lambda^2(L)$.
- Cond 1 of the LCA can be written $\Delta(\Lambda^1(L)) \subset \Lambda^2(L)$.
- What about the 2nd condition, the dual to the Jacobi identity?
- This turns out to be the cocycle condition: $\Delta^2 = 0$.
- Let's see why, and what this means.

Lie Coalgebra (III)

- From now on (and to better tie in to our use case), we'll employ Λ^n to mean $\Lambda^n(L)$, and Λ^1 in lieu of $X = L^*$. Similarly, T^i_j means $T^i_j(L)$.
- The first thing to note is that Δ^2 is a map from $\Lambda^1 \to \Lambda^3$. Like Δ, it takes us beyond the original space, and we now must look more closely at what is happening.
- Unlike $[,]$ on L, the coalgebra operation Δ requires us to work with a graded space of some sort. The question is which one?
- The cojacobi identity involves maps $\Lambda^1 \to \Lambda^1 \otimes \Lambda^1 \otimes \Lambda^1$.
- The right side is T^0_3, not Λ^3. However, as mentioned, the antisymm cond confines it to $\Lambda^3 \subset \Lambda^1 \otimes \Lambda^1 \otimes \Lambda^1$.
- The cojacobi identity can be written in the alternate form $(\Delta \otimes I) \circ \Delta = (I \otimes \Delta) \circ \Delta + (I \otimes \tau) \circ (\Delta \otimes I) \circ \Delta$.
- This form of cond 2 will act as the bridge to the cocycle condition, so we'll call it the "bridge" version.
- We now will do 3 things:
    - Show that this expression is meaningful.
    - Show that the cojacobi identity does indeed correspond to it.
    - Show that it corresponds to the cocycle condition $\Delta^2 = 0$.

Lie Coalgebra (IV)

- First, let's check whether it even makes sense.
  - $\Delta : X \to X \otimes X$, so $(\Delta \otimes I_X) \otimes \Delta$ is a map $X \to X \otimes X \otimes X$.
  - As mentioned, in a given basis we can write $\Delta(e^i) = \Delta^i_{jk} e^j \otimes e^k$, with Δ^i_{jk} antisymm in the lower indices.
  - Then $((\Delta \otimes I_X) \otimes \Delta)(e^i) = \Delta^i_{jk}(\Delta(e^j) \otimes e^k) = \Delta^i_{jk}\Delta^j_{lm} e^l \otimes e^m \otimes e^k$.
  - $(I \otimes \tau)$ just swaps $e^m \leftrightarrow e^k$ — or equivalently, the m, k indices on the Δ's. So we get $\Delta^i_{jk}\Delta^j_{lm} e^l \otimes e^k \otimes e^m$
  - Also $((I_X \otimes \Delta) \otimes \Delta)(e^i) = T^i_{jk}(e^j \otimes \Delta(e^k)) = T^i_{jk} T^k_{lm} e^j \otimes e^l \otimes e^m$.
  - Rearranging the indices to make this clearer (with a $-$ sign for any lower-index swap), we get: $\Delta^i_{jm}\Delta^j_{kl} = -\Delta^i_{jk}\Delta^j_{lm} + \Delta^i_{jl}\Delta^j_{km}$
  - This may be written $\Delta^i_{jk}\Delta^j_{lm} + \Delta^i_{jl}\Delta^j_{mk} + \Delta^i_{jm}\Delta^j_{kl} = 0$ (note that we swapped k and m in the 2nd term, hence no minus sign).
- The bridge expression thus corresponds to $\Delta^i_{jk}\Delta^j_{lm} + \Delta^i_{jl}\Delta^j_{mk} + \Delta^i_{jm}\Delta^j_{kl} = 0$

Lie Coalgebra (V)

- Next, let's consider the cojacobi identity in its original form.
  - We already saw that $((I \otimes \Delta) \circ \Delta)(e^i) = \Delta^i_{jk}\Delta^k_{lm} e^j \otimes e^l \otimes e^m$. In our standardized form, this is $-\Delta^i_{jk}\Delta^j_{lm} e^k \otimes e^l \otimes e^m$.
  - ξ rotates the indices k, l, m once, and ξ^2 does so twice. Adding these, we get: $\Delta^i_{jk}\Delta^j_{lm} + \Delta^i_{jl}\Delta^j_{mk} + \Delta^i_{jm}\Delta^j_{kl} = 0$.
  - This is the same as the bridge expression.
- This may look a lot like the plain old Jacobi identity, and that's because it is identical. This is unsurprising since the information content is the same and we just reversed the arrows. We'll see this more clearly via structure constants shortly. However, we mustn't lose sight of the fact that the maps involved are quite different. Δ and ϕ are maps between different spaces, though related.
- We still must show equivalence to the cocycle condition $\Delta^2 = 0$.
- The problem is that Δ^2 has no meaning. Unlike $[,]$ which keeps us within L, Δ does not keep us within Λ^1, and cannot be composed with itself. It moves us from $\Lambda^1 = L^*$ to $\Lambda^2 = L^* \wedge L^* \subset L^* \otimes L^*$.
- In fact, the only reason it keeps us within Λ^2 instead of $\Lambda^1 \otimes \Lambda^1$ is because of the antisymmetry property $\Delta = -\tau \circ \Delta$.

Lie Coalgebra (VI)

- To have any hope of a relation between the cojacobi identity and $\Delta^2 = 0$, we must move beyond L^*.
- We saw that an anti-deriv on $\Lambda(L)$ is uniquely determined by its action on Λ^0 and Λ^1.
- Let us suppose we have some specified action of Δ on Λ^0 (which is just the ground field or ring). Then the LCA conditions define a unique extension of Δ to any anti-deriv on all of $\Lambda(V)$.
- Note that we do *not* have a LCA on $\Lambda(V^*)$, any more than $\mathcal{L}$ (which we'll see is the unique extension of the LA of v.f.'s to $T(M)$) defines a LA on all of $T(M)$. We simply are taking a particular map $\Delta : \Lambda^1 \to \Lambda^2$, and extending it to the full $\Lambda(L)$.
- This is perhaps clearer if we define a completely independent structure on $\Lambda(L)$, which we'll evocatively call d. It has the following defining properties:
  - d has a specified action on the ground field or ring.
  - d agrees with Δ on Λ^1.
  - d is an anti-derivation on Λ.
- Such a d is unique, if it exists. It therefore contains no additional information beyond that of Δ and the choice of action on the ground field or ring.

Lie Coalgebra (VII)

- Let us now compute d^2. We know how to do this. Let $x, y \in \Lambda^1$.
- $d(x \wedge y) = (dy) \wedge x - x \wedge (dy) = (dx) \wedge y - (dy) \wedge x$, where the sign is unchanged because $\deg(dy) = 2$.
- $de^i = \Delta^i_{jk} e_j \otimes e_k$, because it agrees with Δ on Λ^1.
- Let's rewrite this as $de^i = \frac{1}{2}\Delta^i_{jk}(e^j \wedge e^k)$, where the $1/2$ is because we don't constrain $j < k$ even though we now have a $\wedge$.
- So $d^2 e^i = \frac{1}{2}\Delta^i_{jk}[(de^j) \wedge e^k - (de^k) \wedge e^j] = \Delta^i_{jk}(de_j) \wedge e^k$, where the factor of 2 is because the two terms differ only in index names.
- Notice there is no $d\Delta^i_{jk}$ term, as we've come to expect in DG. This has to do with our view of Λ as an algebra over K rather than $C^\infty(M)$. We'll have a lot more to say about this shortly. For now, we simply observe that it is the source of the apparent discrepancy.

Lie Coalgebra (VIII)

- We can rewrite our expression as $\frac{1}{2}\Delta^i_{jk}\Delta^j_{lm}e^k \wedge e^l \wedge e^m$ (where we passed e^k through $e^l \wedge e^m$, so no sign change).
- Our previous results were expressed in terms of $\otimes$, so let's express everything in terms of $\otimes$ on $\Lambda^1 \otimes \Lambda^1 \otimes \Lambda^1$.
- There are 6 terms, each a permutation.
- Specifically, the wedges resolve to $e^k \otimes e^l \otimes e^m - e^k \otimes e^m \otimes e^l - e^m \otimes e^l \otimes e^k + e^m \otimes e^k \otimes e^l - e^l \otimes e^k \otimes e^m + e^l \otimes e^m \otimes e^k$
- Swapping the l and m in the Δ's just changes sign, so there are pairs of equal terms, and we have 3 distinct terms (and the $\frac{1}{2}$ disappears since we multiply each term by 2).
- We can pick the relevant term from each pair as we wish, and will go with the following (for ease of seeing the result).
- $\Delta^i_{jk}\Delta^j_{lm}[e^k \otimes e^l \otimes e^m - e^l \otimes e^k \otimes e^m + e^m \otimes e^k \otimes e^l]$
- Each sum is over all k, l, m, so we can rewrite this as (swapping k and m in the middle term).
- $[\Delta^i_{jk}\Delta^j_{lm} + \Delta^i_{jl}\Delta^j_{mk} + \Delta^i_{jm}\Delta^j_{kl}]e^k \otimes e^l \otimes e^m$
- The cojacobi identity sets the bracket to 0, so $d^2\Lambda^1 = 0$.
- The converse holds too. $d^2\Lambda^1 = 0$ is equiv to the cojacobi identity.
- $d^2 = 0$ beyond Λ^1 follows from this and the anti-derivation rule.

Relationship between structure constants

- Ex. let's see how the info in $[,]$ and d are related in DG.
- Recall $d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$ for 1-forms.
- Pick local basis fields $\{e_i\}$ for the v.f.'s, with dual basis (for one-forms) $\{e^i\}$, where $e^i(e_j) = \delta^i_j$.
- Note that we don't require a global choice of basis fields, since that would amount to a section of the frame bundle LM, which means LM would be a trivial PB. For our purposes, a local section suffices (and always exists).
- $de^i(e_j, e_k) = e_j(e^i(e_k)) - e_k(e^i(e_j)) - e^i([e_j, e_k])$.
- The 1st term on the right is just $e_j(\delta^i_k)$. In local coords, this is $\partial_i(\delta^i_k)$. But the derivative of a constant fn δ^i_k is 0. So $e_j(\delta^i_k) = 0$. The same is true for the 2nd term.
- We thus have $de^i(e_j, e_k) = -e^i([e_j, e_k])$. But $[e_j, e_k] = c^l_{jk} e_l$ is the def of the LA structure constants (in the module over $C^\infty(M)$ view).
- So $de^i(e_j, e_k) = -c^l_{jk} e^i(e_l) = -c^l_{jk}(\delta^i_l) = -c^i_{jk}$. I.e., d acting on 1-forms just gives us the LA structure constants, with a sign flip.
- The LA and LCA have the same canonical content, as expected.