Duality between d and $\mathcal{L}$ (III)

K.M. Halpern

2021

Summary

- Quick Recap of de Rham Cohomology
- Review of Algebra Concepts
- Graded Algebras
- Tensor Algebra
- Exterior Algebra
- Derivations
- Derivation Uniqueness
- A bit more about the interior product

Quick Recap of de Rham Cohomology (I)

- Integration pairs an n-form with something resembling an n-chain and produces a real number. Specifically, it pairs an element of degree n in the de Rham cochain complex of M with an element of dimension n in the dual to the singular chain complex of M.
- Stokes' thm tells us that DR cohomology of M is homomorphic to AT cohomology of M.
- De Rham's thm tells us it goes both ways, and the two cohomology theories are isomorphic.
- I.e., as far as cohomology goes, the de Rham complex and AT singular cochain complex are interchangeable — as are the de Rham cocomplex and the AT singular chain complex.
- Note: we have not claimed that the de Rham complex itself is isomorphic to the AT singular cochain complex. This would require that $\Lambda^n \approx C^n$ for all n and that $d = \partial^*$ for all n.
- There is something called a "quasi-isomorphism" between the two cochain complexes, which really is just a name for what we described: that the chain map induces a homology isomorphism.

Quick Recap of de Rham Cohomology (II)

- As mentioned, the analogue of the de Rham complex's wedge product in the singular cochain complex is called the "cup" product. There is an algebraic sort-of-isomorphism between the two.
- The dual of the de Rham cochain complex is a chain complex, and is analogous to the singular chain complex in AT in the same sense that the DR complex is analogous to the singular cochain complex. As mentioned, the elements of this DR-dual chain complex can be interpreted as generalized paths. I.e., higher-dimensional analogues of a path which can retrace itself.
- One last comment about the relationship between closed and exact forms in the DR complex. Exact forms always are closed, but closed forms need not be exact. However, the Poincare Lemma tells us they *are* locally exact. If $d\omega = 0$, then around any point $x \in M$ there is a chart U s.t. $\omega = d\alpha$. More precisely, ω is exact in any contractible region. The de Rham cohomology is the obstruction to a global solution in general. It is the bit of the closedness that *cannot* globally be made exact.

Review of Algebra Concepts (I)

- Ring: Two ops: $+, \times$. Abelian Group under $+$, Monoid under $\times$. Mult distributes across $+$.
  - Usually, the term "ring" refers to "ring w/unit" — which has a multiplicative identity 1. We will mean this.
  - Division Ring: A ring (w/unit) in which all nonzero elements have inverses.
  - Commutative Ring: $x \times y = y \times x$.
  - Recall that monoids are associative, so every ring (commutative or not) is associative under both $+$ and $\times$.
- Integral Domain: Commutative ring in which nonzero elements never multiply to 0. Equivalently, $ab = cb$ iff $a = c$ (for $b \neq 0$).
- Field: A commutative division ring. I.e., an abelian grp under $+$, and sans 0 an abelian group under $\times$, and distributive laws hold. Every field is an integral domain. Ex.: $Z_2, \mathbb{R}, \mathbb{C}$. Note that $C^\infty(\mathbb{R})$ is NOT a field.

Review of Algebra Concepts (II)

- (Two-sided) Module over Ring R: An Abelian group M along with scalar mult $(\cdot : R \times M \to M)$ s.t.
  - $\cdot$ distributes over $+$
  - $\cdot$ is associative-like: $(ab) \cdot x = a(b \cdot x)$. I.e., scalar mult and ring mult are compatible.
  - $1 \cdot x = x$. The ring identity is a scalar mult identity as well.
- Vector Space: Module over a Field.
- Algebra over a Commutative Ring R (or Field K): A module over R (or vector space over K) with mult $\times$, s.t.
  - $\times$ distributes over $+$
  - Scalar mult is compatible with $\times$: $(a \cdot v) \times (b \cdot w) = (a \cdot b)(v \times w)$.
  - Note that $\times$ need not have an identity, be invertible, or even be associative. As with rings, we generally mean "algebra w/unit" when we use the term "algebra."
- An assoc algebra can be viewed as a ring of its own w/compatible scalar mult by a distinct commutative ring R (or field K). This is not true of nonassoc algebras, since a ring always is assoc under $\times$.

Graded Algebras (I)

- Chains, cochains, homology, and cohomology were purely algebraic concepts. So are graded rings, graded modules, graded algebras, tensor algebras, exterior algebras, derivations, and anti-derivations, and we'll discuss them in general terms, albeit with an eye toward our goal.
- Graded Ring: A ring which may be written $R = \oplus_{i=1}^{n} A_i$ s.t.
  - Each A_i is an Abelian group
  - The copy of A_i in R is termed of "grade" i.
  - $+$ for the ring is defined in the obvious way as $(x_1 \ldots x_n) + (y_1 \ldots y_n) = (x_1 + y_1, \ldots, x_n + y_n)$ (where $x_i, y_i \in A_i$).
  - $\times$ for the ring obeys $A_i A_j \subseteq A_{i+j}$. Note that A_i here refers to its copy in R, which can be operated on by the ring mult. We've also denoted mult $A_i A_j$ since $A_i \times A_j$ could be confused with the direct product.
  - I.e. elements of grades i, j multiply to elements of grade $i + j$.
  - Recall that for finite n, $\oplus$ is the same as a direct product. So we may think of R as $A_1 \times \cdots \times A_n$, with an additional algebraic structure (internal ring mult) on it.

Graded Algebras (II)

- A "basis element" for an Abelian group just refers to a generator. Any element of R can be written $x = \sum_i x_i$ where $x_i \in A_i$. I.e., x can be expanded as a sum over all basis elements from all A_i, with integral coefficients (no scalar mult, just $nz = z + \cdots + z$ n times).
- Suppose $x = x_1 + x_2$ and $y = y_1 + y_2$ with $x_1, y_1 \in A_1$ and $x_2, y_2 \in A_2$. Then, $x \times y$ is not in any specific grade. $x_1 \times y_1$ sits in grade 2, $x_1 \times y_2 + x_2 \times y_1$ sits in grade 3, and $x_2 \times y_2$ sits in grade 4.
- $x \times y$ is of a pure grade iff x and y both are, at least for the division rings we deal with.
- However, $x + y$ can be of pure grade even if x and/or y isn't. Things can cancel out. Ex. $x_1 + x_2$ and $x_1 - x_2$ both are of mixed grade, but their sum is of grade 1.
- The point is that ops on pure grades result in pure grades, but ops on mixed grades may not (and for $\times$ cannot).

Graded Algebras (III)

- We're interested in graded algebras, so let's proceed in that dir.
- Graded Module M over a graded ring R: A module $M = \oplus M_i$ in the obvious sense, with each M_i a module over (all of) R, and s.t. scalar mult adds the grades appropriately: $R_i \cdot M_j \subseteq M_{i+j}$.
- Often, we deal with graded modules over ungraded rings. Ungraded is the same as trivial grading. In that case, all of R has 0 grade, and scalar mult simply must preserve the grade of each M_i.
- Ditto for a graded v.s. over a field. This is just a graded module over an ungraded ring that happens to be a field.
- Graded Algebra: Viewing A as a ring with outside scalar mult by another (commutative) ring R, we require that
  - A be a graded ring.
  - $R_i \cdot A_j \subseteq A_{i+j}$
- This is compatible with viewing A as a graded module with compatible internal mult (i.e. $A_i A_j \subseteq A_{i+j}$).
- As before, when R is ungraded (ex. a field), we simply can view A as a graded ring with grade-preserving scalar mult.

Graded Algebras (IV)

- Aside: A "superalgebra" is a graded alg with two grades ($A = A_0 \oplus A_1$) and where mult adds grades mod 2. I.e. $A_i A_j \subset A_{(i+j) mod 2}$.
- A graded algebra may seem a bit similar to a cochain complex, and several of the cochain complexes we study are indeed graded algebras as well. However, these are two distinct structures. There are several key differences to keep in mind:
  - A cochain complex has no internal multiplication defined on it. In this, it would be more akin to a graded module.
  - A cochain complex is a disjoint union, not a direct sum. Its elements (i.e. "chains") each reside in one and only one C^i. An element of a graded algebra or graded module can mix grades. Put another way, we cannot add chains from C^i and C^j.
  - A cochain complex has a map d (or ∂^*) between the C^i. A graded algebra has no such structure.
- Because graded algebras which also are chain complexes arise in AT and DG, it turns out to be useful to give them a name. However, the simple presence of two distinct structures doesn't especially empower us beyond their individual properties. For a composite to be useful, the underlying structures must be "compatible" in some sense.

Graded Algebras (V)

- But first a word from our sponsors. From now on, we'll use $x \cdot y$, xy, and $x \times y$ for internal mult. We'll also use $c \cdot x$ or cx for scalar mult, and it will be clear from the context whether scalar mult or internal mult is involved. $x \times y$ (internal mult) vs $S \times S'$ (direct product) will be clear from context as well. For the most part, we'll stick with $\cdot$ for both scalar and internal mult.
- Differential Graded Algebra: A graded algebra A which also is a chain complex, and whose chain complex structure is compatible with the alg structure. Specifically, it has a map $d : A \to A$ s.t.
  - $d^2 = 0$
  - $d(x \cdot y) = (dx) \cdot y + (-1)^{\deg(x)} x \cdot (dy)$ for $x, y \in A$, with $\deg(x)$ denoting the grade of x (or more formally, the deg of homogeneity).
- Both the tensor algebra and the de Rham complex are differential graded algebras.
- As alluded to, the singular cohomology (the H_n, not the sing cochain complex itself) forms a diff graded alg, with the product the "cup product", and d something called the "Bockstein Homomorphism."

Tensor Algebra (I)

- Some words of warning about tensor algebras and exterior algebras.
  - Many treatments, particularly those not related to DG, define tensor algebras solely in terms of contravariant tensors (i.e. tensor products of V's). In DG (ex Kobayashi-Nomizu), we define them as allowing covariant factors (i.e. V^*) too.
  - The transition to exterior algebras differs depending on whether one is dealing with a purely contravariant tensor algebra or a mixed one. However, it can be accomplished with either.
  - We'll start with the purely contravar def and move to the mixed one.
  - We must keep very careful track of whether we are working with a tensor/exterior algebra over a commutative ring or one over a field. Many of the subtleties in going from the algebra at a point on M to that on all of M are associated with this distinction!
  - In particular, some of the objects we encounter may be viewed either as ∞-dim vector spaces over a field or as finite-basis modules over a ring of smooth fns. Many of the apparent discrepancies between the abstract formalism and the practice of DG are the result of this. We will have much more to say about this later.
  - Bear in mind that $\mathbb{R}$ and $\mathbb{C}$ are fields, but $C^\infty(M)$ is a comm ring.

Tensor Algebra (II)

- We'll start with the purely contravar def of a tensor alg.
- Tensor algebra $T(V)$ of v.s. V (over field K):
  - Just the direct sum $\bigoplus_{i=0}^{\infty} V^{\otimes i}$, where $V^{\otimes i}$ means a tensor product $V \otimes \cdots \otimes V$ of i copies of V.
  - We include $T^0 V \equiv K$. This is a 1-dim v.s.. Both $T^0 V$ and K itself are referred to as the "ground field."
  - I.e. $T(V) = K \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \ldots$.
  - The v.s. structure carries over in the obvious way.
  - The internal mult is $t \otimes s$. This takes $T^n V \otimes T^m V \to T^{n+m} V$ via linearity in the obvious way.
- $T(V)$ is an associative algebra.
- $T(V)$ is a graded algebra.
- $T(V)$ is in fact a differential graded algebra, where the relevant d produces a sort of alternating sum which looks a lot like the action of i_X on forms. We won't use this structure.
- Note that, the tensor algebra defined this way *solely* involves upper indices. It is an algebra of contravariant tensors. There are no factors of V^* involved.

Tensor Algebra (III)

- Every grade is a vector subspace, but not every grade is a subalgebra.
- Only $T^0 V = K$ is a subalg ($0 + 0 = 0$, so no grade change under its internal mult). Any higher grade (in particular, the copy of V itself as $T^1 V$) is *not* an algebra or closed under $x \otimes y$.
- An entirely analogous def holds for $T(X)$ where X is a module over some commutative ring R.
- Consider a manifold M of type K ($K = \mathbb{R}, \mathbb{C}$). Let the canonical tangent space be $tm \equiv K^{\dim M}$. Then $T(tm)$ is a graded alg over K.
- The tensor fields on M also form a graded algebra, but as a module over the comm ring $C^\infty(M)$.
  - The basis is that of tm, but the coeffs now are smooth fns on M.
  - I.e., it has the same basis as tm, but is now a module over $C^\infty(M)$.
  - We'll denote it $T(M)$.
  - Note that we cannot claim $T(M)$ has finite "dimension," like tm. Each coeff is a fn. As a v.s. over K, $T(M)$ would have uncountable dimension (countable if M is compact, I think). As a module over $C^\infty(M)$, there is no notion of dimension.
  - We'll have much more to say about such issues later.

Tensor Algebra (IV)

- What we've described so far involves just copies of V. It also is possible to define a tensor algebra over V which involves both copies of V and V^*.
- The machinery is the same.
- Instead of just grade T^i, we have grade $T^i_j = V^{\otimes i} \otimes V^{*\otimes j}$.
- Addition is defined component wise, multiplication is the tensor product (with appropriate distribution of addition across it).
- In this case, the requirement is that $T^i_j \otimes T^k_l \subset T^{i+k}_{j+l}$.
- We now have another op, though: contraction. It is induced by the natural action of V^* on V, and is a generalization of the trace. We'll have more to say about this shortly.
- In principle, we needn't confine ourselves to V and V^*. We could build a tensor algebra from multiple different v.s.'s instead of copies of V or V^*. The only things which would change are the fungibility of certain indices and the absence of a contraction. Duals eat vectors, but a general W would not eat V.

Tensor Algebra (V)

- In DG, when we speak of the Tensor Algebra of V, we mean the mixed one just described, not the pure contravariant one found in many treatments. We'll denote it $T(V)$. Unless otherwise stated, we will mean the mixed algebra by $T(V)$ and $T(M)$.
- A v.s. isomorphism between V and W uniquely extends to one between $T(V)$ and $T(W)$. In fact, the result is even stronger.
- Thm: There is a bijection between the v.s. isomorphisms $V \leftrightarrow W$ and those algebraic isomorphisms $T(V) \leftrightarrow T(W)$ which preserve grade and commute with contractions [Kobayashi pg. 24, prop 2.12].
- Corollary: The group of (linear) automorphisms of V is naturally isomorphic to the group of tensor-algebra automorphisms of $T(V)$ which preserve grade and commute with contractions.
- Everything we discussed about going from $T(tm)$ to $T(M)$ extends to mixed tensors as well.

Exterior Algebra (I)

- There is a way to define this directly from the contravariant tensor algebra, using ideals.
- However, it is much easier to define from the mixed tensor algebra we're using, so we'll do it that way.
- Exterior algebra $\Lambda(V)$ of v.s. V (over field K):
  - $\Lambda^i(V) \equiv \{A(t) | t \in T_i^0\}$, where A is the antisymmetrization operator, and T_i is the grade $(0, i)$ purely covariant tensors.
  - I.e., $\Lambda^i(V)$ consists of all fully antisymmetric fully covariant tensors.
  - $\Lambda(V) \equiv \oplus \Lambda^i(V)$.
  - Each Λ^i is a vector subspace of T_i^0, and inherits addition and scalar mult.
  - The algebra's internal mult is defined via the usual wedge product: $x \wedge y \equiv A(x \otimes y)$ for $x, y \in \Lambda(V)$.

Exterior Algebra (II)

- Note that the internal mult on T is *not* meaningful on Λ, because Λ is not closed under $\otimes$. The tensor product of two fully-antisymmetric tensors is not fully anti-symmetric.
- I.e. Λ is *not* a subalgebra of T, even though it is a vector subspace.
- Λ is a graded algebra.
- Λ is an associative algebra.
- We extend to M in the usual way. $\Lambda(M)$ is a module subspace of $T(M)$, over the commutative ring $C^\infty(M)$.
- Note: When working with $char(K) = 2$, there are some subtleties in the quotienting process by which an exterior alg is derived from its tensor alg. These are evident in the contravariant tensor algebra approach, but are ignored in our approach. The exterior algebra obtained still has the desired antisymmetry properties. However, for $char(K) = 2$, the exterior alg is constrained by more than mere antisymmetry.

Derivations (I)

- Now, let's return to plain old algebra, and define derivations. We'll then move to derivations and anti-derivations on graded algebras.
- Derivation on alg Y over a ring or field K: A map $D : Y \to Y$ s.t.
  - D is linear (under Y's addition, and scalar mult by K)
  - Leibnitz rule: $D(xy) = xD(y) + D(x)y$
- The commutator $(D_1 \circ D_2 - D_2 \circ D_1)$ of two derivations is a derivation.
- Denote by $Der_K(Y)$ the set of all derivations on alg Y over K.
- As an aside, $Der_K(Y)$ is a LA under the derivation commutator.
- Let's briefly consider what the objects we care about really are.
- $\mathcal{L}_V$ is defined on each grade T^i_j of the tensor algebra T individually, but also has the form of a derivation on T as a whole due to its product rule.

Derivations (II)

- However, we have to be careful about the other things we usually think of as derivations. They are not. There are several key points of departure.
  - Some have the form $f(xy) = f(x)y + (-1)^p x f(y)$ for some odd p. d and i_X have this issue.
  - In fact, p could be dependent on the element x. d and i_X have this issue.
  - Some are defined only on a graded algebra as a whole. They change the grade of an element, and individual grades are not closed under them.
    - $\mathcal{L}_X$ has closure for each grade individually.
    - ∇, d, and i_X do not since they change the grade of their argument.
- It is clear that we need something more than a derivation to describe most of the things we think of as differentials or differential-like.
- ∇ is a derivation, but requires a graded structure. i_X and d are "anti-derivations," defined only on a graded algebra as a whole. We'll now define this notion more precisely.

Derivations (III)

- Anti-derivation: Given graded-algebra X, a linear map $D : X \to X$ which takes grade i to grade $i + k$ for some fixed k (i.e. $D(A_i) \subseteq A_{i+k}$), and satisfies $D(xy) = D(x)y + (-1)^{|p||k|} xD(y)$, where p is the grade of x.
- k is termed the "degree" of D. It will be $-1, 0, 1$ for the cases we'll consider. Ex. $k = -1$ for i_X, $k = +1$ for d, and $k = 0$ for $\mathcal{L}_V$ (yielding a plain derivation).
- Note that an anti-derivation on a non-graded algebra technically would be a derivation (if we call the degree 0), but the name is evocative of the opposite. To avoid confusion, we (not the world) will call the case of $D(xy) = D(x)y - xD(y)$ a "skew-derivation" to distinguish the two.
- The point is that D (for $k = \pm 1$) acts like a derivation on even grades, and like a skew-derivation on odd grades.
- Note that "anti-derivation" refers to the totality of D, *not* just the grades with the $-$ sign. I.e., it encompasses both derivation-like and skew-derivation-like behaviors.

Derivation Uniqueness (I)

- There are a number of important uniqueness results which illuminate the situation. Note that they do not guarantee existence.
- Prop 3.2 [KN]: Every derivation D on $T(M)$ has a unique decomposition into $D = \mathcal{L}_X + S$ for some v.f. X and some $(1, 1)$ tensor field S.
- Lemma [KN]: A derivation on $T(M)$ is uniquely determined (if it exists) by its action on T^0 and T^1. I.e., its action on fns and v.f.s.
- Lemma: $\mathcal{L}_X$ is the unique derivation on $T(M)$ s.t. $\mathcal{L}_x f = X(f)$ and $\mathcal{L}_X Y = [X, Y]$.
- Prop 3.8a-c [KN]: Let D and D' be derivations or anti-derivations on $\Lambda(M)$ of degrees k and k' (i.e. D takes p forms to $p + k$ forms).
  - $[D, D']$ is a deriv of deg $k + k'$ if both are derivs.
  - $[D, D']$ is an anti-deriv of deg $k + k'$ if one is a deriv and one is an anti-deriv.
  - $DD' + D'D$ is a deriv of deg $k + k'$ if both are anti-derivs.
- Prop 3.8d [KN]: A deriv or anti-deriv on $\Lambda(M)$ is uniquely determined (if it exists) by its action on 0 forms and 1 forms.

Derivation Uniqueness (II)

- So basically, derivs and anti-derivs on $T(M)$ or $\Lambda(M)$ are uniquely defined by their action on the lowest 2 grades. This doesn't guarantee existence — just uniqueness if it exists.
- Note that uniqueness is indep of the deg of the deriv/anti-deriv. We're not saying there is a unique deriv of a given deg, just a unique one overall. Its degree (if it exists at all) will be obvious from its behavior on the lowest two grades.
- Prop 3.9 [KN]: For every v.f. X, $\mathcal{L}_X$ is the unique deriv of deg 0 (i.e. preserving grade) s.t. $[\mathcal{L}_X, d] = 0$ (where we restrict $\mathcal{L}_X$ to Λ, of course). All such derivs on $\Lambda(M)$ have the form $\mathcal{L}_X$ for some X. I.e., if it commutes with d and has deg 0, it's a Lie Derivative.
- Note the action of $\mathcal{L}_X$ when restricted to $\Lambda(M)$. $\mathcal{L}_X(f) = X(f)$, of course. However, $(L_X(\omega))(Y) = X(\omega(Y)) - \omega([X, Y])$
- i_X is the unique anti-deriv on $\Lambda(M)$ s.t. $i_X f = 0$ and $i_X\omega = \omega(X)$. It has degree -1.
- d is the unique anti-deriv on $\Lambda(M)$ s.t. $(df)(X) = X(f)$ and $(d\omega)(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$. It has degree $+1$.

A bit more about the interior product (I)

- Recall the def of the interior product:
  - Given v.f. X and n-form ω (with $n \geq 1$), the interior prod $i_X\omega$ is the $(n-1)$-form $(i_X\omega)(X_1 \dots X_{n-1}) \equiv A[\omega(X, X_1, \dots, X_{n-1})]$.
  - As before, A is the antisymm operator $A(Z_{i_1 \dots i_n}) \equiv \frac{1}{n!}\sum_{\pi(i_1 \dots i_n)}(-1)^{\pi(i_1 \dots i_n)} Z_{\pi(i_1 \dots i_n)})$. I.e., a sum over all permutation of indices. In this case, we take all perms of $X, X_1 \dots X_{n-1}$. We won't be overly pedantic about this. It's just the obvious way to create a form from a tensor.
  - So $i_X : \Lambda^n \to \Lambda^{n-1}$
  - By convention, we define $i_X f = 0$ for 0-forms.
- i_X has the following properties (easy to show):
  - $i_X i_Y \omega = -i_Y i_X \omega$
  - $i_X i_X \omega = 0$.
  - $i_X(\alpha \wedge \beta) = (i_X\alpha) \wedge \beta + (-1)^p \alpha \wedge (i_X\beta)$ (for α a p-form).
  - $i_{[X,Y]} = [\mathcal{L}_X, i_Y]$
- I.e., Λ is a chain complex under i_X, and i_X is an antiderivation over the exterior algebra.
- The structures are compatible and Λ is a diff graded alg under i_X.

A bit more about the interior product (II)

- Prop: i_X is the unique antiderivation on Λ s.t. $i_X \alpha = \alpha(X)$ for 1-forms.
- Recall that (from its axioms) d is the unique antiderivation on Λ s.t. $df(X) = X(f)$ (i.e. df is the differential).
- Both form mathematical chain complexes (i_X a chain complex, d a cochain complex) on Λ.
- I.e., the interior and exterior maps look an awful lot like duals of some sort, albeit with the same underlying chain groups Λ^n.
- In fact, it may be tempting to think i_X and d are inverses. However, this is not the case. They are related by Cartan's formula, which in turn tells us the real sort of duality we should be looking at.