Duality between d and $\mathcal{L}$ (II)

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2021

Summary

- Integration
- Stokes' Thm and de Rham's Thm
- Some more on Lie Derivatives

Integration (I)

- We've discussed chains and cochains as names for mathematical chains in a chain complex. We've also discussed chain maps, homology, and cohomology.
- But we haven't really talked about how chains are connected to integration.
- Loosely speaking, we know that we integrate n-forms over n-manifolds. An n-manifold is a topological space, and thus has a chain complex. We also have a boundary map in Stokes' thm, so this presumably should be related to the ∂ in the chain complex. But what really is going on?
- Is the manifold a chain or a chain complex or something else altogether?
- Let's consider what integration means in the context of algebraic topology.

Integration (II)

- Recall that a top space gives rise to a chain complex with cells or simplices as the basis elements for the C_i, and boundary maps as ∂_i which obey $\partial^2 = 0$.
- A continuous map between top spaces induces a chain map between these: a set of maps $\alpha_i : C_i \to C_i'$ which form commuting squares with the ∂_i and ∂_i' maps of the two chains.
- A chain complex induces homology groups H_n, a chain map induces homomorphisms $H_n \to H_n'$ (i.e. is covariant), and a SES of chains induces a LES of homology groups.
- In this context, we can construct a cochain complex as the dual of the chain complex for the top space X. Each C^i is the space of linear maps $C_i \to Z$. The ∂'s induce ∂^*'s which go in the opposite direction (i.e. ascending dimension). It is common practice to elevate the coeffs to $\mathbb{R}$, so the C^i are v.s.'s.
- What do the cochains mean in this context? An element of C^i is defined by values on the basis. I.e., it is a set of real values on the simplices or cells which form the basis of C_i. The ∂^* is a linear combo of the values on the boundary. Ex. For a 1-simplex, it would map C^0 to C^1 by assigning a 1-simplex the difference of the values on the endpoint 0-simplices. I.e. a derivative.

Integration (III)

- This sounds nice and taut, but it is overly simplistic.
- It is true that a given top space X regarded as a simplicial complex or CW complex has an associated chain. For simplicial and cellular homology, the basis of C_i is the specified set of simplices or cells of dimension i, and the boundary maps connect them. The resulting space is homeomorphic to X, but has some shortcomings:
  - Suitable (i.e. homeomorphic) simplicial or CW complexes don't exist for most spaces. They are "toy" models.
  - The cochains are linear maps from the simplices to $\mathbb{R}$. These take constant values on each simplex. To get to anything resembling integration of a function on a manifold we would need some sort of limit — perhaps an ever-finer mesh of many simplices.
  - There is no obvious way to incorporate any notion of smoothness. Edges and corners are irrelevant from the standpoint of topology (except in their function as boundaries).
- Consider the torus. It is homeomorphic to a simplicial complex of 18 2-simplices, 27 1-simplices, and 8 0-simplices. This simplicial complex captures the topological notion of continuity, but there is no obvious way to impose a differential structure. It cannot capture the manifold struct of $S^1 \times S^1$ or its local diffeomorphisms to $\mathbb{R}^2$.

Integration (IV)

- The singular chain complex is a generalization which solves these problems.
- Instead of a fixed set of simplices glued together a specific way *as* the top space, we consider a maximal set of continuous maps of canonical simplices *to* the top space.
- Each C_i is all possible cont. maps from a canonical i-simplex to X.
- This space is HUGE, and some heavy machinery is needed to show that the homology is the same as simplicial or cellular homology in the cases where the latter exist. But they are.
- Singular homology solves our problems:
  - Every top space has a singular chain complex and singular homology.
  - Though not obvious at first glance, we now can capture the notion of non-constant functions on simplices. The infinite number of embeddings of a single canonical simplex serve in lieu of an infinitely fine mesh of simplices.
  - We can represent smooth functions as particular cochains in the relevant dual complex.

Integration (V)

- How does the integral play into this?
- We are used to things like $\int_M \omega$, where M is an n-manifold and ω is an n-form.
- We also have seen Stokes' thm in the form $\int_M d\omega = \int_{\partial M} \omega$.
- Strictly speaking, these are not quite right. More precisely, they hide what is really going on from the standpoint of algeb top (AT).
- The correct way to write an integral is $\int_C \omega$, where C is a chain.
- Let's consider this for a moment. We'll allow C to be a chain in the AT sense and ω to be a form in the de Rham sense.
- Bear in mind that the chain complex is not a graded v.s.. An element is a member of one and only one C_i, *not* a linear combo of multiple. There is a well-defined notion of dim (or degree) for a chain.

Integration (VI)

- We may consider $\int$ as a map taking an AT chain C and a de Rham cochain ω and giving us a real number.
- $\int : C_p \times \Lambda^p \to \mathbb{R}$.
- For given p-form ω, $(\int \omega) : C_p \to \mathbb{R}$ is a linear map. If we add regions or loops or whatever, we add integrals.
- We thus have a linear map from $C_p \to \mathbb{R}$, which is just an element of the AT (dual) cochain C^p using $\mathbb{R}$ coefficients.
- I.e., each ω defines an element of the cochain C^p via the $\int$ map. It is easy to see that $\int$ is linear in ω as well.
- Therefore, we may write $\int : \Lambda^p \to C^p$, and we have a v.s. homomorphism from Λ^p to C^p.

Stokes' Thm and de Rham's thm (I)

- The correct form of Stokes' thm is $\int_C d\omega = \int_{\partial C} \omega$.
- We'll mention what C means for manifolds in a moment.
- Stokes' thm tells us that, restricted to closed forms and cycles, $\int$ induces a well-defined map between the cohomology group of the de Rham complex and that derived from the cochain complex.
- I.e., Stokes' thm is a crucial element in showing that the cohomologies via chains and via the de Rham complex are the same.
- How does Stokes' thm do this?
  - $\int$ maps $\Lambda^p \to C^p$. It is a map between cochains (i.e. asc chains), given by $\int(\omega)(C) = \int_C \omega$.
  - Let's check whether it is a (co)chain-map. To avoid confusion, we'll use ∂^* as the dual to ∂, and reserve d for the cochain d of the de Rham complex (i.e. the exterior derivative).
  - In this notation, the req for a (co)chain-map is that $\int \circ d = \partial^* \circ \int$.
  - Recall that $\partial^*(x) = x \circ \partial$ for $x \in C^i$. This is AT.
  - $\int_C d\omega = (\int d\omega)(C) = (\int \circ d)(C)$
  - $\int_{\partial C} \omega = (\int \omega)(\partial C) = (\partial^* \circ \int)(C)$
  - I.e., Stokes' thm is precisely the cochain-map condition.

Stokes' Thm and de Rham's thm (II)

- Integration defines a map from the de Rham complex to the AT cochain, and Stokes' thm tells us that it is a chain map.
- We therefore have induced homomorphisms $H^p_{DR} \to H^p_{AT}$.
- de Rham's thm tells us the converse — and turns these homomorphisms into isomorphisms. It is nontrivial to prove.
- We could go the other way (obviously), in which case the "chains" dual to the de Rham complex are comprised of "submanifolds" of a sort. Such "submanifolds" may run around the same bit of manifold a few times, however. I.e., they are parametrized maps into the manifold. Think of a path integral over a path that wraps around a circle multiple times.
- This is the meaning of AT chains, and what we really integrate over.
- Integration and Stokes' thm operate on submanifolds of this sort. I.e. generalized paths. $\int_M$ means this in a particular context.
- At its heart, Stokes' thm is a simple statement about a single simplex and its boundary! The rest follows from linearity.
- The wedge product on Λ corresponds to the "cup" product for AT cohomology. The latter grants C^* (the AT cochain complex) the structure of a graded algebra.

Stokes' Thm and de Rham's thm (III)

- NOTE: A top space is mapped to a chain complex, not an element in a chain complex. Its topology is encoded in the sizes of the C_i and in the boundary maps.
- NOTE: Integration only applies to manifolds, because the de Rham (differential) complex only exists on them. It critically depends on the local def of its action on 0-forms via the chart coord derivatives. Any well-behaved top space has a singular chain complex, and thus a cochain complex, homology, and cohomology. Only for manifolds is a de Rham cohomology defined, and in that case it is equal to the AT cohomology.
- NOTE: Integration may be defined for arbitrary forms and arbitrary elements of the chain complex, and Stokes' thm applies to these as well. The cochain map $\int$ contains more information than Stokes' thm. The latter just tells us that it *is* a cochain map.
- We also can think of the content of Stokes' thm (for closed forms and cycles) as:
  - Closed forms have 0 integral over boundaries. $\int_M d\omega = 0 = \int_{\partial M} \omega$ for $d\omega = 0$.
  - Exact forms have 0 integral over cycles. $\int_M \omega = \int_M d\alpha = \int_{\partial M} \alpha = 0$ for $d\omega = \alpha$ and cycle $\partial M = 0$.

A bit more about Lie Derivatives (I)

- Let's now return to Lie Derivatives for a bit and consider the intuition behind them.
- Any v.f. V on M generates a flow $\phi : \mathbb{R} \times M \to M$. We may write $\phi_V(p, t)$ as the point which p flows to after time t under V.
- Locally, the flow is the solution to: $\frac{\partial x_i}{\partial t} = V^i(x)$, where $V^i(x)$ is the coefficient of $V(x)$ in the induced basis ∂_i.
- We thus have a one-parameter family of autodiffeomorphisms on M.
- We may write these as $\Phi_V(t)$, where (for given t) $\Phi_V(t) : M \to M$ is just the curried form of ϕ, and $\phi_V : \mathbb{R} \to Diff(M)$. I.e., $\Phi_V(t)(p) = \phi(p, t)$, and it is our time translation operator.
- Under the action of $\Phi_V(t)$ for some given t, a point p maps to $p' \equiv \Phi_V(t)(p)$.
- Any form or tensor field T has some values at p and p'. We may pull back $T(p')$ along $\Phi_V(t)$ to some $T^*(p')$ at p . We then have two tensors at p, and may compare these.

A bit more about Lie Derivatives (II)

- Because we are dealing with diffeomorphisms, this extends without issue to fields (regardless of whether T is a form, v.f., or mixed tensor field).
- We could go the other way too (pushing forward $T(p)$ to p' and comparing two tensors at p'), but this is less convenient since we must track the movement $p \to p'$ as well. Keeping everything at p is very convenient because everything remains in the same fiber.
- Let's fix V and t, and write $\phi(p) \equiv \Phi_V(t)(p)$ (i.e. p' from before).
- Denote by $T^*(p)$ the pull-back of T along $\Phi_V(t)$ from $\phi(p)$ to p. Note that we now are writing $T^*(p)$, not $T^*(p')$ for the same thing!
- Define $\Delta(p) \equiv T(p) - T^*(p)$. At each point, it is the difference between $T(p)$ and the pull back of $T(\phi(p))$ to p. This is well-defined because both $T(p)$ and $T^*(p)$ are tensors at p.
- All of these constructs are implicit functions of V and t.
- Now take the limit as $t \to 0$.
- In local coords, we can linearize this to $\Phi_V(\epsilon) = I + \frac{d\Phi_V(t)}{dt}|_{t=0}\epsilon$, where I is the local identity map for the chart. There is no global equivalent, of course — since we don't have addition or scalar multiplication on M itself!

A bit more about Lie Derivatives (III)

- We know what the right side is from the earlier formula for ϕ. It's just the coefficients V^i of V in the induced coordinate basis.
- Unlike for Φ_V itself, we *can* directly linearize Δ and take the limit, since it exists in a linear vector space!
- The Lie Derivative is just $\mathcal{L}_V(t) = \lim_{t \to 0} \frac{\Delta(t)}{t}$. This also can be viewed as $\frac{dT^*(p)}{dt}|_{t=0}$ for any T and p.
- Hmm... a one parameter family of diffeomorphisms per M? Infinitesimal generators for it? If this seems familiar, it is.
- We have a LG of autodiffeomorphisms on V. Equivalently, we have the action of some group G on M.
- V is a generator for this group. I.e., an element of the LA.
- In fact, the LA is the linear space of v.f.s on M, endowed with the v.f.-commutator as Lie Bracket. It is a LA commutator since it obeys the Jacobi identity.
- What is the associated LG? It turns out to be all of $Diff(M)$. Note that infinite-dim LA's are not the LA's of infinite-dim LG's in general, but in this case it is. However, there remain 2 shortcomings (due to being infinite-dim):
  - There are problems if e^{tV} is not defined at some finite t
  - The exponential map can fail to be a local diffeomorphism at identity. [My note: presumably, these are limiting issues?]

A bit more about Lie Derivatives (IV)

- We have a LG G acting on M, and it has induced transforms on tensor fields, etc. It may be tempting to think the Lie Derivative acts on ABs. It doesn't, but let's see why.
  - Let P be the PB over M with (very large) group $G = Diff(M)$.
  - We may try to consider P_F, the AB with $F = T^n_m$ or $F = \Lambda^k$.
  - Such fibers aren't arbitrary. They relate to the struct of M. In fact, the corresponding FBs are often termed the "natural bundles" of M.
  - We now may think: ok, there is a natural induced action (the pull-back action), so we have an action on F.
  - However, we don't. We have a pull-back $\Phi^*_V(t)$ which acts on tensor fields. This is *not* an action on the space of tensors at a given point (i.e. the fiber over a point in M). It requires information from elsewhere. We must obtain the value of T at a different point in M.
  - $\Phi^*_V(t)$ acts on sections of the tensor bundles. It is not meaningful to speak of its action on the fibers themselves.
  - This is not surprising. An action on the fiber would have to be linear.

A bit more about Lie Derivatives (V)

- So what is $\mathcal{L}_V$?
- $\mathcal{L}_V$ is the LA action of V on $\Gamma(T)$. The relevant LG acts on the space of sections (of any given tensor bundle) over M, and $\mathcal{L}_V$ is a generator of the corresponding flow on that space of sections. I.e., $\mathcal{L}_V$ is to $\Gamma(T^k_j M)$ what V is to M. [My Q: Can it therefore be mapped to a v.f. on $\Gamma(T^k_j M)$]?
- The correct way to think of $\mathcal{L}_V$ is not via ABs and actions — but in terms of derivations and extensions to the tensor algebra.