INFLUENCE IN BLOCK VOTING SYSTEMS

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Abstract. Motivated by the examples of the electoral college and Supreme Court, we consider the behavior of binary voting systems in which votes are cast in blocks of possibly unequal size. We wish to determine the relevance of participants as a function of the distribution of block sizes. To this end, we provide an intuitive definition of "influence" as the likelihood that a participant can change the outcome of an election subject to certain strong assumptions. After examining a few simple examples, the structural implications of block voting, and the analytical properties of influence, we apply our definition to the specific case of electoral voting. We compute state influences for the current electorate using a Monte Carlo analysis, and observe that the influence of states is not a linear function of their electoral allocation.

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1. Introduction

1.1. **Motivation.** In the closely contested presidential elections of 2000 and 2004 media pundits variously ascribed the ability to "decide" the outcome to several "swing" states. Similar claims have been made about certain Supreme Court justices. Such assertions are necessarily ill-defined in the absence of a precise definition of what it means for a participant to decide an election.

In a different vein, consider the viewpoint of an individual voter. A voter in an area that strongly favors one party may consider his vote irrelevant, regardless of his preference. He may consider voting a waste of time[1].

Similarly, most small investors do not participate in proxy votes for shares they own. Even if the decisions involved are important to them, they believe that the outcome is determined by a few large shareholders. Without judging these views, we may observe that in any given voting situation each of us has an intuitive sense of whether our vote really matters.

There is no universal way to evaluate the relevance of a vote without knowledge of the behavior of one's fellow voters. However we may arrive at a meaningful notion of "influence" as a participant's ability to decide the outcome of an election. Furthermore, we may distinguish the components that are structural, based on the allocation of votes and the underlying independence of decisions, from those based on a presumptive probability measure over voting patterns.

To illustrate the conceptual ambiguity in a blind voting situation, let us consider a few examples.

1.2. **Case : Equal Votes in a Clubhouse.** Consider two clubhouse elections held by 9 individuals with a single vote each (similar to a non-deliberative Supreme Court).

(1) $(8, 1)$ vote. No single individual could have changed the election outcome by changing his vote.
(2) $(5, 4)$ vote. Any of the 5 individuals in the majority could have changed the outcome had they voted differently.

Structurally, each individual has the same influence. This must be true mathematically because they are fungible; there is no preferred participant. While it is pointless to ask who could have affected the election after the fact, most participants would feel that they had greater influence in the second case. This is particularly true of the majority voters in the second example.

1.3. **Case : Popular Election in a Party Stronghold.** Consider a popular election in which there are two parties (A and B), each individual casts a single vote, the election is decided by a simple majority, and a certain set of individuals always vote for a specific party. If the number of such party-line voters for party A is a strong plurality of the total vote, an individual voter may feel that his vote doesn't matter. Regardless of how he casts it, the election outcome is unaffected.

[1]We will not consider abstention as an option. That is a game theory problem, in which individuals have a large negative utility if less than the desired majority shows up but a small negative utility if they unnecessarily add their vote to an already sufficient majority.

1.4. **Case : Presidential Election from State Standpoint.** Consider a simplified presidential election in which there are two parties, and states commit votes to either in blocks. Because the states command varying numbers of votes, the question arises whether certain states may never influence the outcome of an election. For example, consider two scenarios involving four states with the specified allocation of votes:

(1) $(7, 5, 3, 2)$: Treating each state as an independent binary choice, there are 16 possible voting patterns. There exist scenarios where any one of the states could have changed the outcome by flipping its vote.
(2) $(3, 3, 3, 2)$: Again, there are 16 possible voting patterns. In none of them could the state with 2 votes have changed the outcome by flipping its vote. Assuming the other 3 participants always voted, that state could have refrained from voting. It has no influence.

While it is unlikely that a large set of blocks such as the electorate would combinatorically exclude any member, we cannot assume that the influence of a block is proportional to its size. We will see that it is not.

1.5. **Case : Supreme Court Split Along Idealogical Lines.** Each member of the Supreme Court has an equal vote. In practice, the Supreme Court often votes along ideological lines. We thus hear of a "swing vote", a specific individual who is not strictly aligned with either block. In this case, there are effectively 3 participants with unequal numbers of votes. For a swing vote to be possible, the blocks would have to be $(4, 4, 1)$. Then, there are two cases:

(1) The blocks vote together and the individual cannot affect the outcome.
(2) The blocks oppose one another and the individual always affects the outcome.

The only reason the individual exerts such influence is because of the block alignments. If the two blocks always oppose one another, they are superfluous and the individual should be the only participant. While this may be the outcome in practice, his disproportionate influence arises from the presence of the blocks and the choice of the individuals in those blocks to oppose one another. Any individual in those blocks could wield the same influence or destroy the swing voter's status by removing their alignment. The influence of the swing vote arises from incidental rather than structural alignment of the other votes[2]. Nonetheless, it it is a consequence of the number of true independent decisions and their respective weights.

1.6. **Hints at Influence.** In each of the examples provided, we have a sense of whether an individual has influence. It depends on both the structure of the election and the outcome (or equivalently, the behavior of other voters). An ex-post-facto measure of influence is pointless because the realized outcome is unalterable. If one voter could really change their vote then so could everyone else, amounting to a simple revote. We are not concerned whether an individual could have changed the outcome of a particular election. Rather, we are interested in understanding the likelihood that his vote could decide the outcome of a future election. There

[2]In some sense, it is immaterial whether the blocks are structural or the result of underlying behavioral correlations as long as they are invariable. We are not examining voting strategies, so the knowledge the swing voter has is irrelevant; he always votes his conscience. We will return to this point in section 2.2.

are two possibilities: a quantity that is purely based on the structure of the system or a statistic over the space of voting scenarios using an assumed probability distribution. The salient features are that

(1) In any given predictive scenario, an individual can influence the election if a change in his behavior could alter the outcome.
(2) Many individuals may be able to influence a given election.
(3) The critical factor in determining influence is the real block voting behavior. It is the number of independent decisions that are made and the number of votes assigned each that matter[3].

There are different types of admissible behaviors we may want to consider. The simplest would be for a participant to change his vote. This is the definition we employ herein. It implicitly assumes that the participant is forced to vote and could potentially have voted the other way. Another possibility would be to ask whether the presence of the participant in the election could make a difference. This assumes that given a set of candidates or choices, each voter would deterministically vote a specific way. The question then becomes whether his abstention would have altered the outcome. In section 4.9 we note that such a system is equivalent to the one we consider. There are other possibilities as well, and we discuss some means of generalization in section 6.

2. Definition of Block Voting System

2.1. **Framework.** Given these examples as motivation, we define the concrete problem under examination.

Definition 1. *A* **Block Voting System** B *is a set of* $N \geq 3$ *participants*[4] *and a vector* $n = (n_1 \dots n_N)$ *of positive integer block sizes representing the relative weights assigned to the participants' votes. For convenience, we list the* n *from greatest to smallest:* $n_i \geq n_{i+1}$.

Definition 2. *An* **Election** v **by Block Voting System** B *is a vector* $v = (v_1 \dots v_n)$ *of binary choices* $v_i \in \{+1, -1\}$ *representing a specific vote by each participant. The procedural assumptions associated with the election are:*

(1) *Each individual makes a binary choice* (± 1).
(2) *Each participant's block of votes is allocated as a whole based on their choice. The block is not divisible.*
(3) *No participant refrains from voting.*
(4) *Each participant makes an independent decision, unaffected by the votes of others*[5].

We refer to this as an **Election**, with an implicit reference to the associated B. To clarify our definitions, consider a simplified US electoral vote[6]. The "Block Voting System" would consist of the electoral structure and rules as well as the specific choice of candidates before the electorate that year. An "Election" would be a specific hypothetical set of choices by the electors.

[3] See section 2.2 for further discussion of this.
[4] We will use the term "individual" and "participant" interchangeably.
[5] The idea of independence is one we will clarify.
[6] The election by the electoral college which formally selects the President.

Definition 3. *The* **Count** $X(v)$ **of an Election** v *is the total sum of votes* $X(v) = \sum_i n_i \cdot v_i$.

Definition 4. *The* **Outcome** $O(v)$ **of an Election** v *is the sign of* $X(v)$. *In the case of a tie* $X(v) = 0$, *we define* $O(v) = 0$.

A few notes on our definitions:

- We do not assume blind and simultaneous voting. Although that is the model for most of the systems we examine, it is an unnecessary restriction. Our other assumptions address the situations which would merit its use.
- There can be no gaming. There is no direct way to game a binary system, but one could conceive of a scenario in which an individual would prefer to vote with the winning party in order to reap the benefits of loyalty. Such a case is excluded by the requirement that participants' votes are unaffected by the votes of others.
- There can be no collusion. We do not concern ourselves with the formation of each individual's preference. Debate and negotiation are possible prior to a given vote. However, we require independence. For example if a party boss could command his followers' votes, the appropriate specification of the system would treat them as a single aggregate block. We will discuss the notion of independence further in section 2.2.
- In some sense, the requirement of indivisibility is unnecessary. In a binary election, division of one's block could only serve to reduce an individual's influence. If the participant is an aggregate of independent sub-blocks (for example, if a 3 vote block is really 3 independent voters), then we have misspecified the block structure. In no properly specified binary block voting system would a division of votes be fruitful[7].
- We have excluded abstention, but could model it in a variety of ways. For example, we could consider the set of voting systems that are subsets of the one we wish to examine. However, this is beyond our current scope. We are not considering whether someone should stay home because they think their vote won't matter. We are attempting to measure whether it actually could matter.

2.2. **Independence.** One of the core requirements of our block voting system is that the blocks vote independently. This is a point that requires some elaboration. In modeling a real system, we may wish to consider both structural and statistical behavior. For example, the formal rules of a voting system may require certain blocks of indivisible votes. However, in practice certain participants may always vote the same way. We could ignore this and treat them as independent because they theoretically are; their past behavior does not guarantee they will vote coincidentally in the future. However we may also wish to look at this from a statistical standpoint. In that case, rather than merely reflecting structural rules of the real system, the blocks could be chosen to represent our best understanding of the true independent decisions being made. For example, if a set of 20 structural blocks vote based on 3 underlying criteria, then there are only 3 independent decisions being made. We could employ a factor model to construct effective blocks. The utility of the resulting system in analyzing future elections is subject to all the

[7]However, a divisible vote could be used to mimic abstention.

assumptions of statistical inference as well as our choice of underlying factors. As will be seen, our definition of Influence requires an assumed probability distribution anyway. The imposition of a statistical block structure would constitute an implicit conditioning of the probability distribution over elections.

It is important to note that free will is not at issue. If two participants view the world exactly the same way, they would always vote accordingly under our assumptions[8]. For example, suppose they would vote for whichever party endorses issue Z. If only one of parties A and B do so in any election, and it is party A with probability 0.8, we could conclude from historical analysis that they each would vote for party A with probability 0.8. If we treat the two voters as independent, then we allow them to vote against one another - which they never do. They are not colluding, but represent a single decision point. On the other hand, if we combine them into a single block that votes for A with probability 0.8 we discount the possibility that they ever could differ. Proper factor analysis would give us a much better sense of the underlying decisions. It is important to keep in mind the distinction between inviolable structural blocks and statistical blocks which are subject to various assumptions.

3. Structural Considerations

3.1. **Basic Questions.** There are two interesting questions that immediately arise in any block voting system:

- Can a given voter ever affect the outcome of an election?
- Are there voters who cannot affect the outcome of any election?

These questions have answers that are purely structural and do not depend on any assumptions about the likely outcome of an election.

3.2. **Irrelevance.**

Definition 5. *For a given block voting system B, we denote by S_B the sample space of all possible election vectors v. Its cardinality is 2^N.*

Definition 6. *We say that* **participant i has influence on election** v *if changing the sign of v_i alone would result in a change in $O(v)$ (including to or from a tie).*

Proposition 1. *For a block voting system B, participant i has influence on a given election v iff $0 \leq v_i \cdot X(v) \leq 2n_i$. Equivalently, $|\sum_{j \neq i} n_j \cdot v_j| \leq n_i$.*

Proof. If participant i changes his vote then the sum swings by $-2v_i \cdot n_i$. The only way this can influence the outcome (including to or from a tie) is if

- $v_i > 0$ and $0 \leq X(v) \leq 2n_i$ or
- $v_i < 0$ and $-2n_i \leq X(v) \leq 0$

These may be combined into $0 \leq v_i \cdot X(v) \leq 2n_i$. Noting that $v_i^2 = 1$ and subtracting n_i from all three parts of the inequality we get $-n_i \leq v_i \cdot \sum_{j \neq i} n_j \cdot v_j \leq n_i$, which yields the 2^{nd} form. □

Proposition 2. *For a block voting system B and a given election v:*

1. $X(-v) = -X(v)$
2. *Participant i has influence on v iff he has influence on $-v$.*

[8]It could be interesting to examine the role of noise in this.

(3) *Participant i has influence on $v = (v_1 \dots v_n)$ iff he has influence on $v' = (v_1 \dots v_{i-1}, -v_i, v_{i+1} \cdots v_n)$. If he can change the outcome by flipping his vote one way, then he can do so by flipping it back too.*
(4) *The smallest block cannot influence an election in which all larger blocks agree.*

Proof. :

(1) $X(-v) = \sum n_j \cdot (-v_j) = -\sum n_j \cdot v_j = -X(v)$
(2) This follows from the invariance of $|\sum_{j \neq i} n_j \cdot v_j| \leq n_i$ under $v \to -v$.
(3) The condition $|\sum_{j \neq i} n_j \cdot v_j| \leq n_i$ is independent of any reference to v_i.
(4) Let all the larger votes be $+1$. For the smallest block to have influence, we require $\sum\limits_{j<N} n_j \leq n_N$. There are at least 3 blocks and $n_1 \geq n_2 \geq n_N$, so $n_1 + n_2 > n_N$ and n_N has no influence on this election.

□

Definition 7. *We say that* **participant** i **is relevant** *if there exists an election $v \in S_B$ on which he has influence. The participant is* **irrelevant** *if he is not relevant.*

Proposition 3. *For a block voting system* **B**, *participant* **i** *is relevant iff the following equivalent conditions hold:*

(1) *There exists an election v with $v_i > 0$ and $0 \leq X(v) \leq 2n_i$.*
(2) *There exists an election v such that $0 \leq \sum\limits_{j \neq i} (n_j \cdot v_j) \leq n_i$.*

Proof. Let participant i be relevant, and let v be an election on which he has influence. From Prop 2 we see that i will have influence on $-v$ as well, so choose whichever has $v_i > 0$ or whichever has $\sum_{j \neq i}(n_j \cdot v_j) \geq 0$ respectively. Going the other way, if v obeys either of the two conditions, it trivially satisfies the corresponding condition in Prop 1. □

Definition 8. *We define a* **relevance vector** r *associated with block voting system B. It is of length N and each element is 1 if the corresponding participant is relevant and 0 otherwise.*

Both of our questions can be answered by an exhaustive examination of every election vector v. This is conceptually straightforward, if computationally tedious[9].

Definition 9. *The* **Closest Count** $C(B)$ **of a Block Voting System** *is* $\min\limits_{v \in S_B} |X(v)|$. *It is the closest that any election can come to a tie*[10].

Remark. *We can restate the definition of relevance in terms of the Closest Count. For block voting system B and participant i, define the reduced system B' which is just B excluding i. Participant i is relevant in B iff $|C(B')| \leq n_i$.*

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[9]Note that for the determination of r_i, it suffices to sample all $v \in S_{B'}$ with B' the reduced B with n_i absent. However, the result is clearly the same.

[10]Note that from Prop 2 there exists a v with $X(v) = C(B)$ positive.

3.3. **Basic Properties of Relevance.** Before proceeding, let us examine some properties of our definition of relevance.

Proposition 4. *For a given block voting system B of size N, with block sizes $\mathbf{n}$, and relevance vector $\mathbf{r}$, the following hold:*

1. *$\mathbf{r}$ is invariant under division or multiplication of $\mathbf{n}$ by a positive integer. We need only consider systems where there is no common divisor to the $\mathbf{n}$.*
2. *Identical blocks are interchangeable. If $n_i = n_j$ then $r_i = r_j$.*
3. *If there are any irrelevant blocks, then $X(v) \neq 0$ for all elections v.*
4. *If a block is irrelevant then so are all smaller blocks. If $r_i = 0$ then $r_j = 0$ for all $n_j \leq n_i$.*
5. *If a block is relevant then so are all larger blocks. If $r_i = 1$ then $r_j = 1$ for all $n_j \geq n_i$.*
6. *The largest block is always relevant*[11].
7. *Only the largest block is relevant if it is larger than the sum of all smaller blocks. If $n_1 > \sum_{j=2}^{N} n_j$ then $r_1 = 1$ and $r_i = 0$ for $i > 1$.*

Proof. :

1. The condition to have influence $|\sum_{j\neq i} n_j \cdot v_j| \leq n_i$ is invariant under multiplication or division of n by a constant. Because we demand that our n_i be positive integers, scaling only makes sense if a is a positive integer or $\frac{1}{a}$ is a common integral divisor of all the n_i.
2. Suppose i is relevant. By Prop 3 there exists an election v such that $0 \leq n_j \cdot v_j + \sum_{k\neq i,j} n_k \cdot v_k \leq n_i$. Define an election v' which is the same as v but with $v'_i = v_j$ and $v'_j = v_i$. Noting that $n_j = n_i$ here, $0 \leq n_i \cdot v'_i + \sum_{k\neq i,j} n_k \cdot v'_k \leq n_j$. Relevance of i implies relevance of j.
3. Every block has influence on an election with $X(v) = 0$ because any vote flip will move it to a nonzero value, changing $O(v)$.
4. Suppose block i is irrelevant but a smaller block j is relevant. There exists an election v with $v_j > 0$ such that $0 \leq X(v) \leq 2n_j$. But $n_j < n_i$, so $0 \leq X(v) < 2n_i$. If $v_i > 0$, let $v' = v$. Otherwise, define v' to be v with $v'_i = +1$ and $v'_j = -1$. Then, $X(v') = X(v) + 2n_i - 2n_j$. As $n_i > n_j$, we have $0 \leq X(v) < X(v')$. On the other hand $X(v) \leq 2n_j$ so $X(v) + 2n_i - 2n_j \leq 2n_i$. Combining these, $0 \leq X(v) + 2n_i - 2n_j \leq 2n_i$ and $0 \leq X(v') \leq 2n_i$. Block i has influence on v', contradicting our assumption.
5. This follows from the previous result. Suppose j is relevant. If i is irrelevant and $n_i \geq n_j$ then so are all smaller blocks, including j.
6. Choose an election $v^{(0)}$ with $X(v^{(0)}) > 0$. We construct a sequence of elections $v^{(0)}, v^{(1)}, v^{(2)}, \ldots$ in the following manner: given $v^{(i)}$ we flip the vote of block $i+1$ to get $v^{(i+1)}$. Eventually, the outcome must change and $X(v^{(i)}) \leq 0$ for some i. If this did not occur, then we would end up with $X(-v^{(0)}) > 0$, violating our assumption. The block whose flip changed the outcome between elections $v^{(i)}$ and $v^{(i+1)}$ has influence on election $v^{(i)}$ and is therefore relevant. All larger blocks are relevant too, including the largest block.

[11]Note that this doesn't mean the largest participant can change the outcome of a particular election. For example, if all blocks are small and an election has a wide margin then no single participant could have changed the outcome by voting differently.

(7) For $i > 1$, $C(B')$ is easy to compute. Block 1 (the largest one) is present in B' so all the other blocks must oppose it. However $\sum_{j>1, j\neq i}^{N} n_j < \sum_{j>1}^{N} n_j < n_1$, so $C(B') = n_1 - \sum_{j>1, j\neq i}^{N} n_j = n_i + (n_1 - \sum_{j>1}^{N} n_j) > n_i$. No i has relevance except for $i = 1$.

□

Some notes:

- Because identical blocks are fungible, we refer to blocks as i or n_i unless there is an ambiguity.
- Prop 4 sec 3,4 imply that the relevance vector $\mathbf{r}$ must be a sequence of 1's followed by a sequence of 0's (where we order from largest block to smallest as mentioned). There must be at least one 1 present.
- A block voting system is entirely determined by the number of blocks of each size. Denote that set c, with $c_i \geq 0$ counting the number of blocks of size i and $\sum c_i = N$.
- The number of relevant blocks in the relevance vector (the number of 1's in the sequence) is a function of $\mathbf{r}$, and therefore characteristic of B. Denote it N_R.
- It may be useful to consider B_R, the system we get by removing all irrelevant blocks. That this is meaningful will be demonstrated below in Prop 6.
- We can define two other characteristics of B. Both provide definitions of the smallest block that can add relevance. We define n_R to be the smallest block which would be relevant in the new system (B, n_R) if added to B. Define q_B to be the smallest block which would be relevant if added to the reduced subsystem B_R. Note that if n_i is the largest irrelevant block in B, then $q_B \geq n_i$. We will discuss the effect of adding and subtracting blocks in section 3.5. It is easy to see that $n_R = C(B)$.

3.4. **Enumeration of Block Voting Systems.** Prop 4 provides us with two symmetries of the relevance vector: fungibility of identical blocks and scale invariance. From the standpoint of relevance, a distinct block voting system is an equivalence class under those symmetries. It is natural to ask how many such distinct block voting systems there are. We first analyze this without regard to scale invariance.

Proposition 5. *:*

(1) *The number of systems with N participants and from $1 \dots m$ votes each is* $\binom{N+m-1}{m-1}$

(2) *The number of systems with from $3 \dots N$ participants and from $1 \dots m$ votes each is* $\binom{N+m}{m} - 1 - m - \frac{m\cdot(m+1)}{2}$

Proof. :

(1) Let x_i $(i = 1 \dots m)$ be the number of blocks of size i. We want the number of order-dependent ways to choose the x_i so that $\sum_{i=1}^{m} x_i = N$. This is a standard combinatoric result $\binom{N+m-1}{m-1}$.

(2) We use the combinatoric identity (see [1]) $\sum_{k=m-1}^{N+m-1} \binom{k}{m-1} = \binom{N+m}{m}$ and explicitly subtract the values 1 for $N = 0$, m for $N = 1$ and $\frac{m\cdot(m+1)}{2}$ for $N = 2$.

□

We list some typical values of these in Tables 1 and 2.

TABLE 1. Number of Systems with N Blocks and from $1 \ldots m$ Votes Each

N	m=2	m=3	m=4	m=5	m=6	m=7	m=8	m=9	m=10	m=11	m=12	m=13	m=14
3	4	10	20	35	56	84	120	165	220	286	364	455	560
4	5	15	35	70	126	210	330	495	715	1001	1365	1820	2380
5	6	21	56	126	252	462	792	1287	2002	3003	4368	6188	8568
6	7	28	84	210	462	924	1716	3003	5005	8008	12376	18564	27132
7	8	36	120	330	792	1716	3432	6435	11440	19448	31824	50388	77520
8	9	45	165	495	1287	3003	6435	12870	24310	43758	75582	125970	203490
9	10	55	220	715	2002	5005	11440	24310	48620	92378	167960	293930	497420
10	11	66	286	1001	3003	8008	19448	43758	92378	184756	352716	646646	1144066

TABLE 2. Number of Systems with from $3 \ldots N$ Blocks and from $1 \ldots m$ Votes Each

N	m=2	m=3	m=4	m=5	m=6	m=7	m=8	m=9	m=10	m=11	m=12	m=13	m=14
3	4	10	20	35	56	84	120	165	220	286	364	455	560
4	9	25	55	105	182	294	450	660	935	1287	1729	2275	2940
5	15	46	111	231	434	756	1242	1947	2937	4290	6097	8463	11508
6	22	74	195	441	896	1680	2958	4950	7942	12298	18473	27027	38640
7	30	110	315	771	1688	3396	6390	11385	19382	31746	50297	77415	116160
8	39	155	480	1266	2975	6399	12825	24255	43692	75504	125879	203385	319650
9	49	210	700	1981	4977	11404	24265	48565	92312	167882	293839	497315	817070
10	60	276	986	2982	7980	19412	43713	92323	184690	352638	646555	1143961	1961136

Also of interest are the analogous counts when we allocate a total set of m votes among the blocks. In that case, we require $\sum n_i = m$ (where $m \geq N$) instead of allowing each $n_i \in 1 \ldots m$. In number theory, the partition function $p(m)$ is the number of order-independent sets of positive integers that sum to m, and represents the quantity we seek[12]. Tables 3 and 4 list some typical values of these.

TABLE 3. Number of Systems with N Blocks with a Total of m Votes

N	m=2	m=3	m=4	m=5	m=6	m=7	m=8	m=9	m=10	m=11	m=12	m=13	m=14	m=15
3	—	1	1	2	3	4	5	7	8	10	12	14	16	19
4	—	—	1	1	2	3	5	6	9	11	15	18	23	27
5	—	—	—	1	1	2	3	5	7	10	13	18	23	30
6	—	—	—	—	1	1	2	3	5	7	11	14	20	26
7	—	—	—	—	—	1	1	2	3	5	7	11	15	21
8	—	—	—	—	—	—	1	1	2	3	5	7	11	15
9	—	—	—	—	—	—	—	1	1	2	3	5	7	11
10	—	—	—	—	—	—	—	—	1	1	2	3	5	7
11	—	—	—	—	—	—	—	—	—	1	1	2	3	5
12	—	—	—	—	—	—	—	—	—	—	1	1	2	3
13	—	—	—	—	—	—	—	—	—	—	—	1	1	2
14	—	—	—	—	—	—	—	—	—	—	—	—	1	1
15	—	—	—	—	—	—	—	—	—	—	—	—	—	1

Tables 5 and 6 provide the cumulative results when we also account for scale invariance. The effect is small. As the use of this symmetry greatly increases computational complexity and provides little reduction in sampling, we ignore it in our calculations.

[12]It is not difficult to verify from direct computation that our results match. For example, $p(15) = 176$. Our calculated result for $n \geq 3$ is 168. We add 1 sum of one integer and 7 sums of two integers to obtain 176. See [4] for properties and a table of $p(m)$.

TABLE 4. Number of Systems with from $3 \ldots N$ Blocks with a Total of m Votes

N	m=2	m=3	m=4	m=5	m=6	m=7	m=8	m=9	m=10	m=11	m=12	m=13	m=14	m=15
3	—	1	1	2	3	4	5	7	8	10	12	14	16	19
4	—	—	2	3	5	7	10	13	17	21	27	32	39	46
5	—	—	—	4	6	9	13	18	24	31	40	50	62	76
6	—	—	—	—	7	10	15	21	29	38	51	64	82	102
7	—	—	—	—	—	11	16	23	32	43	58	75	97	123
8	—	—	—	—	—	—	17	24	34	46	63	82	108	138
9	—	—	—	—	—	—	—	25	35	48	66	87	115	149
10	—	—	—	—	—	—	—	—	36	49	68	90	120	156
11	—	—	—	—	—	—	—	—	—	50	69	92	123	161
12	—	—	—	—	—	—	—	—	—	—	70	93	125	164
13	—	—	—	—	—	—	—	—	—	—	—	94	126	166
14	—	—	—	—	—	—	—	—	—	—	—	—	127	167
15	—	—	—	—	—	—	—	—	—	—	—	—	—	168

TABLE 5. Number of Systems with from $3 \ldots N$ Blocks and from $1 \ldots m$ Votes Each (excluding multiples)

N	m=2	m=3	m=4	m=5	m=6	m=7	m=8	m=9	m=10	m=11	m=12	m=13	m=14
3	3	8	15	29	42	69	95	134	172	237	287	377	452
4	7	21	44	92	148	258	384	578	798	1148	1490	2034	2582
5	12	40	93	210	373	692	1113	1787	2648	3998	5549	7912	10626
6	18	66	169	411	800	1580	2737	4677	7409	11761	17378	25928	36743
7	25	100	280	731	1548	3251	6040	10955	18476	30835	48289	75402	112419
8	33	143	435	1215	2781	6199	12300	23614	42238	74044	122418	199918	312732
9	42	196	644	1918	4718	11138	23509	47648	90079	165642	288155	491624	804917
10	52	260	918	2906	7644	19068	42659	91053	181380	349320	637581	1134979	1940678

TABLE 6. Number of Systems with from $3 \ldots N$ Blocks with a Total of m Votes (excluding multiples)

N	m=2	m=3	m=4	m=5	m=6	m=7	m=8	m=9	m=10	m=11	m=12	m=13	m=14	m=15
3	—	1	1	2	2	4	4	6	6	10	8	14	12	16
4	—	—	2	3	4	7	8	12	14	21	20	32	32	42
5	—	—	—	4	5	9	11	17	20	31	32	50	53	71
6	—	—	—	—	6	10	13	20	25	38	42	64	72	97
7	—	—	—	—	—	11	14	22	28	43	49	75	86	118
8	—	—	—	—	—	—	15	23	30	46	54	82	97	133
9	—	—	—	—	—	—	—	24	31	48	57	87	104	144
10	—	—	—	—	—	—	—	—	32	49	59	90	109	151
11	—	—	—	—	—	—	—	—	—	50	60	92	112	156
12	—	—	—	—	—	—	—	—	—	—	61	93	114	159
13	—	—	—	—	—	—	—	—	—	—	—	94	115	161
14	—	—	—	—	—	—	—	—	—	—	—	—	116	162
15	—	—	—	—	—	—	—	—	—	—	—	—	—	163

3.5. **Addition and Subtraction of Blocks.** In order for our notion of irrelevance to make sense, we must show that the removal of an irrelevant block doesn't make other irrelevant blocks relevant. In the following, we must make a careful distinction. When we add or subtract a block, we change the voting system. In speaking of the relevance of a block, we must specify whether this is in the original or modified system, as it may differ. We use the notation $B+n_i$ to mean the system obtained from the union of B and n_i and $B-n_i$ to mean the system obtained by removing the block n_i. Note that if multiple blocks of the same size exist in B, we are only removing one (the i^{th} participant). Technically, B is sorted and the indices may change through this process. For simplicity, we will assume that any block is added or removed off the end and that the indices of other blocks are unchanged. Our meaning will be clear.

Proposition 6. *Given a block voting system B in which participant i is irrelevant, the omission of that participant does not affect the rest of the relevance vector. If we construct a block voting system $B' = B - n_i$ which is the same as B except that it excludes participant i, then the relevance vector for B' is the same as that for B but with the i^{th} element removed. The converse holds too; the addition of a participant n_i who is irrelevant in the new voting system $B' = B + n_i$ does not affect the relevance of any other blocks.*

Proof. There are four cases to consider:

- Suppose we add a participant n_j to B, and he proves irrelevant in the new system $B' = B + n_j$. If n_i is relevant in B, then there exists v with $v_i > 0$ such that $0 \leq X(v) \leq 2n_i$. The effect of adding a vote v_j is to change $X(v)$ to $X(v) \pm n_j$. If $X(v) = 0$ or $X(v) - n_j \leq 0$ then n_j would have influence on $v' = (v, -1)$ and our assumption would be violated. However, $X(v) - n_j < X(v) \leq 2n_i$. So, in the new system we have $0 \leq X(v') \leq 2n_i$ with $v' = (v, -1)$ and $v'_i > 0$. A relevant block in B remains relevant in B' under addition of a block that is irrelevant in B'.
- Suppose a block n_j is irrelevant in B. Let us subtract it to obtain $B' = B - n_j$. Let block n_i be irrelevant in B but relevant in B'. Then, we could add n_j back to B' to get B. But we already know that n_j is irrelevant in B, so the addition of an irrelevant block would have rendered a previously relevant one irrelevant - contradicting our prior result. The subtraction of an irrelevant block of B cannot make any other irrelevant blocks of B relevant in B'.
- Let n_j be irrelevant in B. If n_i is relevant in B, then there exists v with $-n_i \leq \sum\limits_{k \neq i} n_k \cdot v_k \leq n_i$. Also, $n_i > n_j$. Defining $S = \sum\limits_{k \neq i,j} n_k \cdot v_k$, we can write this as $-n_i \leq S + n_j \cdot v_j \leq n_i$. From Prop 2, we may choose whichever of v or $-v$ satisfies $S + n_j \cdot v_j \geq 0$ and separately choose $v_i > 0$. We have $0 \leq S + n_j \cdot v_j \leq n_i$. Suppose we remove block j from B to get $B' = B - n_j$. Define the reduced election $v' \in S_{B'}$ as v without v_j. If $v_j > 0$, then $-n_i < -n_j \leq S \leq n_i - n_j < n_i$ and n_i has influence on v' in B'. If $v_j < 0$, then $n_j \leq S \leq n_i + n_j$. Suppose $S > n_i$. Then, $0 < S - n_i \leq n_j$ which means that n_j has influence on election u in B where u is v but with $u_i = -1$. This would violate the assumption that n_j is irrelevant. Therefore $S \leq n_i$ and $0 < n_j \leq S \leq n_i$ so n_i has influence on v' in B'. As it has influence in either case, n_i is relevant in B'. The subtraction of an irrelevant block of B cannot make a relevant block of B irrelevant in B'.
- Let n_i be irrelevant in B. If we add a block n_j which is irrelevant in $B' = B + n_j$ and n_i were relevant in B', then we could subtract n_j (already known to be irrelevant in B') to get B. However, we have already shown that this could not convert the block n_i from relevance in B' to irrelevance in B. An irrelevant block in B remains irrelevant in B' under addition of a block that is irrelevant in B'.

□

This tells us that by its absence, an irrelevant participant cannot empower other irrelevant participants (or disempower relevant ones). If this were not the case, then irrelevance would not be as meaningful because omission could indirectly affect the outcome of an election. Any or all irrelevant participants can be ignored without affecting the result of any possible election[13]. They truly have no influence. As mentioned, we must be careful (particularly in addition) to specify relevance with respect to the original or modified systems. For example, suppose we start with a block of size 10. If we add 10 blocks of size 1, they all become relevant upon the addition of the 10^{th}, even though they were irrelevant up to that point. When we discuss adding an irrelevant participant, we do not mean adding another instance of an existing irrelevant block. Rather, we mean a block that would be irrelevant in the new scheme. In our example, the addition would work 9 times. However, we would not consider the addition of a 10^{th} block of size 1 to be the addition of an irrelevant block because in the resulting scheme the block would not be irrelevant.

We also can show the following:

Proposition 7. *The addition of a block n_i to B can never make a previously relevant larger or equal block irrelevant in $B + n_i$.*

Proof. Consider addition of n_i. If n_i is relevant in the new system, then all larger (or equal) blocks must be relevant, so any that were already relevant remain so. If n_i is irrelevant in the new system, then we have shown it's addition cannot affect the relevance of existing blocks. □

Note that the addition of a relevant block to a system may render smaller previously relevant blocks irrelevant and the subtraction of a relevant block from a system may render previously irrelevant blocks relevant. In summary, the only operations that may affect the relevance of existing or remaining blocks are:

- A relevant block may become irrelevant if we add a strictly larger block. The new block will be relevant. Ex. $(1, 1, 1) \to (4, 1, 1, 1)$
- An irrelevant block will become relevant if we add a smaller or equal relevant block (ex. $(4, 2, 1) \to (4, 2, 1, 1)$).
- An irrelevant block may become relevant if we add a strictly larger relevant block (ex. $(4, 1, 1) \to (4, 2, 1, 1)$ but not $(5, 3, 1) \to (5, 4, 3, 1)$).
- A relevant block may become irrelevant if we subtract any relevant block. (ex. $(6, 4, 1, 1) \to (6, 4, 1)$ but not $(6, 2, 2, 2) \to (2, 2, 2)$).
- An irrelevant block may become relevant if we subtract a (necessarily larger) relevant block. ex. $(6, 1, 1, 1) \to (1, 1, 1)$.

Also, we may ask what happens if we add a block that is of the same size as a relevant one in the old system.

Proposition 8. *If we add a block n_j to a system B and $n_j \geq n_i$ for some n_i relevant in B, then n_j will be relevant in B'.*

Proof. As n_i is relevant in B, we have $-n_i \leq \sum_{(k \neq i) \in B} n_k \cdot v_k \leq n_i$ for some $v \in S_B$. We may choose v such that the sum is ≥ 0 without loss of generality. Consider the election $v' = (v, -1)$ in the new system (-1 is the vote of n_j). $0 \leq \sum_{(k \neq i) \in B} n_k \cdot v_k \leq n_i$, so $-n_i \leq \sum_{(k \neq i) \in B} n_k \cdot v_k - n_i \leq 0$. But $-n_j \leq -n_i$ and $0 \leq n_j$, so $-n_j \leq \sum_{(k \neq j) \in B'} n_k \cdot v'_k \leq n_j$ and n_j has influence on v' in B'. □

[13]This speaks to their impact on the election, and has nothing to do with actual abstention - which we forbade.

The effect of addition or subtraction of blocks is summarized in Table 7. The columns are operations (addition or subtraction) involving a block n_i with specified relevance in B or B', the rows are the type of blocks on which we wish to understand the effect (specified by relevance and relation in size to n_i), and the values are the resulting relevances of those blocks in the new system B'. The letters R and I denote relevance and irrelevance. Cases where either is possible are denoted RI and accompanied by an example in which the value changes between B and B'. NA is used for cases which are meaningless or impossible.

Table 7. Effect of Addition or Subtraction of Blocks on Relevance

Affected Block	$B' = B + n_i$ n_i rel in B' or B	$B' = B - n_i$ n_i rel in B	$B' = B + n_i$ n_i irrel in B'	$B' = B - n_i$ n_i irrel in B
$n_j < n_i$ rel in B	RI $(1,1,1) \to (4,1,1,1)$	RI $(6,4,1,1) \to (6,1,1)$	R	NA
$n_j = n_i$ rel in B	R	RI $(6,4,1,1) \to (6,4,1)$	R	NA
$n_j > n_i$ rel in B	R	RI $(6,4,1,1) \to (6,4,1)$	R	R
$n_j < n_i$ irrel in B	RI $(4,1,1) \to (4,2,1,1)$	RI $(5,1,1,1) \to (1,1,1)$	I	I
$n_j = n_i$ irrel in B	R	NA	I	I
$n_j > n_i$ irrel in B	R	NA	I	I

3.6. **Merging and Splitting Blocks.** We can also say a little about what happens to relevance if we attempt to merge or split blocks.

If we have incorrectly specified a block and need to split it into subblocks, we can be certain of the following:

Proposition 9. *Suppose we split a block n_i in B into q pieces $m_1 \ldots m_q$ and denote the resulting system of $N + q - 1$ blocks B'.*

(1) *If n_i is irrelevant in B then each of the subblocks are irrelevant in B'*
(2) *If n_i is relevant in B then at least one*[14] *of the subblocks is relevant in B'.*

Proof.

(1) Consider the reduced system $B' = B - n_i$. We are given that n_i is irrelevant in B, so $n_i < C(B')$ and $C(B') > 1$. For any such election $v \in S_{B'}$, we know that $|X(v)| \geq C(B')$. Prop 2 tells us we may restrict ourselves to v with $X(v) > 0$ without loss of generality. In the system $B' + m_1 \ldots m_q$, denote by $u_1 \ldots u_q$ any set of additional votes from the new blocks. The smallest $X(v + u)$ is when all the u are negative. But $X(v) - \sum m_i = X(v) - n_i \geq C(B') - n_i > 0$. So, there is no combination of u that changes the outcome of any election v in the combined system. The m are irrelevant.
(2) n_i is relevant in B, so it has influence on an election v. As before, we can assume $X(v) \geq 0$. Define an election v' in the new system $(B - n_i + m_1 \ldots m_q)$ as $v'_j = v_j$ for the persistent blocks (other than i) and $v'_j = v_i$ for every subblock of n_i. The sum is unchanged: $X(v) - n_i \cdot v_i + \sum_{j=1}^{q} v_i \cdot m_i = X(v)$. Starting with the largest subblock, flip the signs of the corresponding v'_j from v_i to $-v_i$ until one of them changes $O(v)$. We cannot flip all of them without changing it because that would be equivalent to flipping v_i in B, and we chose v such that n_i has influence on it.

∎

[14]An example where some of the subblocks are irrelevant is $(5,4,4) \to (4,4,4,1)$.

If we wish to merge blocks to more accurately reflect the set of truly independent decisions in a system we may be sure that:

Proposition 10. *If we merge two blocks n_i and n_j of B into a single block of B' and either n_i or n_j is relevant in B, the combined block is relevant in B'.*

Proof. Let n_i be relevant. There is an election v on which n_i has influence. We can choose the sign of v so that $v_j > 0$. Then, $-n_i \leq \sum_{k \neq i,j} n_k \cdot v_k + n_j \leq n_i$. Subtracting n_j we have $-n_i - n_j \leq \sum_{k \neq i,j} n_k \cdot v_k \leq n_i - n_j < n_i + n_j$ and the choice of v for $k \neq i, j$ is an election on which the combined block has influence in B'. □

The following is a direct consequence:

Proposition 11. *The sum of the irrelevant blocks in a system is less than the smallest relevant block.*

Proof. Let the sum of the irrelevant blocks $m_1 \cdots m_q$ be $M = \sum m_i \geq n_i$ where n_i is the smallest relevant block. If we consider the reduced system B_R and add a single block of size M to it to get B'_R, Prop 8 tells us that M will be relevant in B'_R. Next, split M into blocks of size $m_1 \cdots m_q$. Prop 9 tells us that at least one of these must be relevant. However the resulting system is B in which $m_1 \cdots m_q$ were irrelevant by assumption. Therefore, $M < n_i$. □

3.7. **Expansion or Contraction of Blocks.** Another interesting question is what happens if we increase or decrease the size of a given block. As with addition and subtraction of blocks, there isn't much we can say in general - otherwise any problem would reduce to a simple sequence of such operations - but we can be certain of the following:

Proposition 12.

(1) *A relevant block remains relevant if we increase its size.*
(2) *An irrelevant block remains irrelevant if we decrease its size.*
(3) *If we increase the size of a relevant block n_i by 1, all larger blocks $n_j > n_i$ remain relevant.*
(4) *If we increase the size of an irrelevant block by 1, all relevant blocks remain relevant.*
(5) *If we decrease the size of an irrelevant block, the relevances of all other blocks are unaffected.*

Proof.

(1) This can be considered two operations: adding a block of size 1 and merging the new block with the relevant block n_i. By Props 7 and 10, the block's relevance is preserved under both operations. We repeat as many times as needed to get to the new size.
(2) To get from n_i to $n_i - m$, split n_i into two blocks: $n_i - m$ and m. Both are irrelevant under Prop 9. By Prop 6, we can then remove the irrelevant block m without affecting any other block's relevance. The remaining $n_i - m$ block is irrelevant.
(3) As we just demonstrated, n_i remains relevant when it becomes $n_i + 1$. By Prop 4, any larger blocks $n_j \geq n_i + 1$ are therefore relevant in the new system too.

(4) This is two operations. First, we remove the irrelevant block n_i. Then, we add a new block of size $n_i + 1$. Whether or not this is relevant in the new system, all previously relevant blocks have $n_j \geq n_i + 1$ and therefore remain relevant in the new system under Prop 8.
(5) Subtract that block from the system. Then add back a block with the new smaller size. It is smaller, so it must be irrelevant and cannot affect the relevance of any other blocks.

□

Table 8 summarizes the effect of expanding or contracting a block. It employs the same conventions as Table 7. The columns are operations involving a block n_i with specified relevance in B, the rows are the types of blocks which may be affected, and the values are the resulting relevance of those blocks in the new system B'. As before, R and I denote relevance and irrelevance. Cases where either is possible are denoted RI and accompanied by an example in which the value changes between B and B'. NA is used for cases which are meaningless or impossible. Note that the last row is the effect on n_i itself.

Table 8. Effect of an Increment or Decrement in Block Size on Relevance

Affected Block	$n_i \to n_i + 1$ n_i rel in B	$n_i \to n_i - 1$ n_i rel in B	$n_i \to n_i + 1$ n_i irrel in B	$n_i \to n_i - 1$ n_i irrel in B
$n_j < n_i$ rel in B	RI $(4,4,3,1) \to (4,4,4,1)$	RI $(8,5,3) \to (8,4,3)$	NA	NA
$n_j = n_i$ rel in B	RI $(4,4,3,3) \to (4,4,4,3)$	RI $(8,4,4) \to (8,4,3)$	NA	NA
$n_j > n_i$ rel in B	R	RI $(8,5,3) \to (8,5,2)$	R	R
$n_j < n_i$ irrel in B	RI $(4,4,4,3) \to (5,4,4,3)$	RI $(4,4,4,3) \to (4,4,3,3)$	RI $(7,4,2) \to (7,5,2)$	I
$n_j = n_i$ irrel in B	NA	NA	RI $(7,3,3) \to (7,4,3)$	I
$n_j > n_i$ irrel in B	NA	NA	RI $(8,4,3) \to (8,4,4)$	I
n_i itself	R	RI $(4,4,4,4) \to (4,4,4,3)$	RI $(8,4,3) \to (8,4,4)$	I

3.8. **Examples of Irrelevance.** Table 9 lists the block voting systems B which have irrelevant participants for $N = 4, 5$ and $n_i \leq 5$, excluding trivial cases in which one block dominates[15].

3.9. **Irrelevance Statistics.** We would like to know the frequency of occurrence of irrelevance as a function of system size. This can be measured in several ways, as described below. In Tables 10 and 11, we compute statistics as a function of the number of blocks N, respectively restricting ourselves to a maximum individual block size $n_i \leq 10$ or to a maximum sum of block sizes $\sum n_i \leq 75$. These results are derived from a full examination of all elections. N is the number of blocks, "Systems" is the number of Block Voting Systems, and "Scenarios" is the number of Block Voting System/Election cases examined. The various statistics we compute are:

- "FracSystems": The fraction of block voting systems which have at least one irrelevant block.
- "FracBlocks": The fraction of blocks which are irrelevant. To get this, we divide the sum of the numbers of irrelevant blocks in all systems by the sum of the numbers of blocks in all systems.

[15]We don't include $N = 3$ because the only such systems with irrelevance have $n_1 > n_2 + n_3$.

TABLE 9. Small N Irrelevance Cases

N	n_R	N_R	Sum	B	Irrel
4	1	3	7	(2, 2, 2, 1)	(1)
4	1	3	9	(3, 3, 2, 1)	(1)
4	2	3	10	(3, 3, 3, 1)	(1)
4	1	3	11	(3, 3, 3, 2)	(2)
4	1	3	11	(4, 3, 3, 1)	(1)
4	1	3	11	(4, 4, 2, 1)	(1)
4	2	3	12	(4, 4, 3, 1)	(1)
4	1	3	13	(4, 4, 3, 2)	(2)
4	3	3	13	(4, 4, 4, 1)	(1)
4	2	3	14	(4, 4, 4, 2)	(2)
4	1	3	15	(4, 4, 4, 3)	(3)
4	1	3	13	(5, 4, 3, 1)	(1)
4	2	3	14	(5, 4, 4, 1)	(1)
4	1	3	15	(5, 4, 4, 2)	(2)
4	1	3	13	(5, 5, 2, 1)	(1)
4	2	3	14	(5, 5, 3, 1)	(1)
4	1	3	15	(5, 5, 3, 2)	(2)
4	3	3	15	(5, 5, 4, 1)	(1)
4	2	3	16	(5, 5, 4, 2)	(2)
4	1	3	17	(5, 5, 4, 3)	(3)
4	4	3	16	(5, 5, 5, 1)	(1)
4	3	3	17	(5, 5, 5, 2)	(2)
4	2	3	18	(5, 5, 5, 3)	(3)
4	1	3	19	(5, 5, 5, 4)	(4)
5	1	3	11	(3, 3, 3, 1, 1)	(1, 1)
5	1	4	11	(4, 2, 2, 2, 1)	(1)
5	1	3	13	(4, 4, 3, 1, 1)	(1, 1)
5	2	3	14	(4, 4, 4, 1, 1)	(1, 1)
5	1	3	15	(4, 4, 4, 2, 1)	(2, 1)
5	1	4	13	(5, 3, 2, 2, 1)	(1)
5	1	4	15	(5, 3, 3, 3, 1)	(1)
5	1	3	15	(5, 4, 4, 1, 1)	(1, 1)
5	1	3	15	(5, 5, 3, 1, 1)	(1, 1)
5	2	3	16	(5, 5, 4, 1, 1)	(1, 1)
5	1	3	17	(5, 5, 4, 2, 1)	(2, 1)
5	3	3	17	(5, 5, 5, 1, 1)	(1, 1)
5	2	3	18	(5, 5, 5, 2, 1)	(2, 1)
5	1	3	19	(5, 5, 5, 2, 2)	(2, 2)
5	1	3	19	(5, 5, 5, 3, 1)	(3, 1)

- "FracVotes": The fraction of votes which are irrelevant. This is a good apple to apple comparison. We divide the sum of the numbers of votes of all irrelevant blocks in all systems by the sum of the votes of all blocks in all systems.
- "SmallBlockFrac": This is the fraction of scenarios in which the smallest block is relevant but does not have influence on the election under consideration. Stated differently, for a system in which all blocks are relevant, this is the likelihood that a (uniformly) randomly sampled election would demonstrate that relevance. It will prove useful in determining our confidence in random sampling[16].

None of these measures represent a real probability that any given block will be irrelevant, as that is dependent on the individual system at hand. However, they do give us a sense of the prevalence of irrelevance as N grows. We expect that for randomly selected[17] large systems, the likelihood of "accidental" irrelevance is low.

[16]The values of 0.5 and 0.25 in Tables 11 and 10 deserve explanation. For a 3 block system, there are 8 elections $v \in S_B$. Block 3 is only relevant if $n_3 \geq |n_1 - n_2|$. If it is relevant, then it has influence on any election in which the two larger blocks oppose one another. This happens 50% of the time. A similar explanation holds for the 0.25 result.

[17]For choice constrained by the maximum block size m, we select a uniformly distributed random integer from $1 \ldots m$ for each block and sort the result. For choice constrained to a fixed overall sum m, we begin by assigning one vote to each block and then assign each of the remaining $N - m$ votes to one of the N blocks by choosing a uniformly distributed random integer from

Unless we specifically engineer a system to exhibit it, we are unlikely to observe irrelevance.

Table 10. Frequency of Occurrence of Irrelevance as a Function of System Size with Max Block Size $m \leq 10$

N	Systems	Scenarios	FracSystems	FracBlocks	FracVotes	SmallBlockFrac
3	220	1760	0.318182	0.212121	0.101928	0.500000
4	715	11440	0.502098	0.170280	0.075334	0.295646
5	2002	64064	0.240759	0.092108	0.032150	0.250905
6	5005	320320	0.226773	0.057110	0.018957	0.159625
7	11440	1464320	0.107168	0.028297	0.008158	0.148754
8	24310	6223360	0.080543	0.015554	0.004509	0.113455
9	48620	24893440	0.038132	0.007551	0.002010	0.108435
10	92378	94595072	0.028048	0.004237	0.001156	0.092591
11	167960	343982080	0.014396	0.002192	0.000559	0.088821
12	293930	1203937280	0.010969	0.001321	0.000345	0.080172
13	497420	4074864640	0.006200	0.000743	0.000183	0.076931

Table 11. Frequency of Occurrence of Irrelevance as a Function of System Size with Sum of Block Sizes $m \leq 75$

N	Systems	Scenarios	FracSystems	FracBlocks	FracVotes	SmallBlockFrac
3	469	3752	0.729211	0.486141	0.244691	0.500000
4	3042	48672	0.730769	0.422830	0.197804	0.250000
5	12470	399040	0.628869	0.336231	0.143432	0.161247
6	36308	2323712	0.516388	0.248770	0.097448	0.098173
7	81612	10446336	0.377653	0.170453	0.063515	0.067659
8	150042	38410752	0.240379	0.108765	0.040905	0.049637
9	235899	120780288	0.133214	0.066691	0.026692	0.040485
10	327748	335613952	0.073538	0.042310	0.018115	0.035785
11	413112	846053376	0.044673	0.028744	0.012870	0.033408
12	481769	1973325824	0.030108	0.020762	0.009531	0.032003
13	528245	4327383040	0.021791	0.015671	0.007291	0.031185

As is evident from Tables 10 and 11, the frequency of irrelevance - by any of our measures - rapidly declines[18] with N.

3.10. **Large N and small block sizes.**

One special case deserves mention. Consider a system with large N, but small $\max(n_i)$. If there are many small blocks, we intuitively expect few examples of irrelevance. Recall our notation c_i for the number of blocks with size i, and let m denote the largest admissible block size. For a given system B, specified by a set of c_i, let us define an election v in the following way: for each c_i that is even we assign an equal number of $+1$ and -1 votes and for each c_i that is odd, we do so with $c_i - 1$ of the blocks and choose $+1$ for the odd block out[19]. Denoting the set of odd cases R, we have $X(v) = \sum_{i \in R} i$.

$1 \ldots N$. Both methods impose a uniform distribution over order-dependent sequences (which are then sorted) rather than order-independent ones. Order-independent sequences are important for counting, and expedite computation, but an order-dependent sampling reflects real systems more closely. Although the members of our mathematical system may be fungible, the behavior of components of a real system likely will not be. This will especially be evident when we compare their historical behavior and require an assumed probability distribution. At this point, however, the distinction is not important.

[18]There is a slight non-monotonicity in the case of $N = 4$ in the "FracSystems" metric. This is not implausible; it is entirely reasonable for the low N cases to be dominated by a few obvious behaviors rather than statistics.

[19]We could bring $X(v)$ closer to 0 by alternating the signs of the odd blocks out or being clever in some other way, but this is unnecessary for the point we are making.

A necessary (but not sufficient) condition for B to have irrelevant blocks is that[20] $2c_1 < X(v)-1$. Otherwise, every block of size 1 would be relevant because we could choose an election v' by starting with v and flipping votes of blocks with $n_i = 1$ and $v_i = +1$ (there are either $\frac{c_1}{2}$ or $\frac{c_1+1}{2}$ of them in v depending on whether c_1 is even or odd) until $X(v)$ becomes non-positive. The last block whose vote was flipped changed $O(v)$, has influence on v', and is relevant - as must be all equal or larger blocks. Therefore, our inequality must hold if the single-vote blocks are irrelevant.

Suppose that the system B was chosen randomly by specifying N and choosing each block size from $1 \dots m$ with uniform probability. The probability that any given block size $1 \leq i \leq m$ is unrepresented ($c_i = 0$) is $(1 - \frac{1}{m})^N$. From the binomial distribution, $\langle c_i \rangle = \frac{N}{m}$ and $\sigma_{c_i} = \sqrt{\frac{N}{m}(1 - \frac{1}{m})}$ for all i. Moreover, the probability that each c_i is even or odd is $\frac{1}{2}$. In the worst case scenario, every c_i is odd and $X(v) = \sum_1^m i = \frac{m(m+1)}{2}$. For large N, the probability $P(c_1 < \frac{m(m+1)}{2} - 1)$ approaches the normal CDF $\Phi\left(\frac{(\frac{m(m+1)}{2}-1)-\frac{N}{m}}{\sqrt{\frac{N}{m}(1-\frac{1}{m})}}\right)$. This in turn approaches $\Phi\left(-\sqrt{\frac{N}{m-1}}\right)$ for large $N \gg m^3$. For example, for $N = 500$ and $m = 5$, the binomial result is $P \sim 10^{-30}$. That is the probability that a necessary condition will be met, an upper bound on the fraction of systems displaying irrelevance. It is clear that for $N \gg m$, the incidence of irrelevance is very small.

3.11. **Confidence of Irrelevance in Sampling.** Establishing the relevance of a block is easy, as it only requires the detection of a single election on which that block has influence. However, without an exhaustive search we cannot be certain of irrelevance. There are some tricks that simplify matters. From Prop 4 we know that the identification of any relevant block indicates relevance of all blocks of greater or equal size[21]. This expedites the identification of relevant blocks. As the sampling progresses, we need only test the remaining smaller blocks.

Suppose that we have tested k samples and have not found block i to be relevant. What is our confidence that it is irrelevant? Again, from Prop 4 if any block is irrelevant then the smallest block is as well. We therefore tackle the slightly less general question: what is our confidence that the smallest block is irrelevant? This is equivalent to ascertaining our confidence that the system as a whole has any irrelevant blocks, one of the two questions we posed earlier. Suppose that the smallest block is relevant and the fraction of elections on which it has influence is f. Sampling with replacement, the probability that we would not encounter such an election in k samples is $P = (1 - f)^k$. For a fairly small sample size of $k = 10^6$ elections, $P < 1\%$ if $f > 4.6 \times 10^{-6}$. From Tables 10 and 11, there appears to be a gradual decline in f (the "SmallBlockFrac" statistic) for high N. This lends plausibility to the argument that $f > 4.6 \times 10^{-6}$ for most reasonable values of N we may consider[22]. For comparison, with $f = 0.01$ we would have $P \sim 10^{-4365}$.

[20] We use $X(v) - 1$ in case R includes 1.

[21] We also know that the largest block is relevant, so we needn't check it.

[22] One possible exception to this analysis would be a very large system (for example an entire population of voters) with varying small block sizes - such as those discussed in Section 3.10.

4. Influence

4.1. **Definition of Influence.** Although we can deduce irrelevance from the structure of a block voting system, the likelihood that a large, naturally arising system would happen to have irrelevant participants is small (except where obvious and accepted). We want a definition of influence that gives us a sense of the relative say that relevant participants have. Such a definition cannot be purely structural, and must depend on a measure over S_B, the space of elections. The case of irrelevance suggests a natural definition of influence.

Definition 10. *Given a probability measure P on S_B, the* **Influence of participant i in B over P**, *denoted I_i, is the probability that he can affect the outcome of an election by changing his vote. Defining the reduced system B_{-i} as B without block i, and using the appropriate reduced probability distribution $P_{-i}(v \in S_{B_{-i}}) \equiv P((v, v_i = +1) \in S_B) + P((v, v_i = -1) \in S_B)$, we can express this as*

$$I_i = \sum_{v \in S_{B_{-i}} \text{ s.t.} |\sum_j n_j \cdot v_j| \leq n_i} P_{-i}(v) \tag{1}$$

Definition 11. *The* **Influence vector** $I(P, B)$ *associated with system B and probability measure P over S_B is the vector of participant influences I_i.*

The relative Influences of participants are determined by the ratios of their I_i's. Because Influence has an interpretation as a probability, we prefer to work with it directly, rather than with relative Influences.

It proves convenient to express the Influences in terms of $v \in S_B$ and P instead of using a distinct reduced system for each block. The following result accomplishes this:

Proposition 13.

$$I_i = \sum_{v \in S_B \text{ s.t.} |\sum_{j \neq i} n_j \cdot v_j| \leq n_i} P(v) \tag{2}$$

Proof. Consider an election $v' \in S_{B_{-i}}$ in the domain of the sum in equation 1. It satisfies $|\sum_j n_j \cdot v_j| \leq n_i$ and has reduced probability $P_{-i}(v')$. The two corresponding election vectors in S_B are $v_+ = (v', +1)$ and $v_- = (v', -1)$, identical to v' in all common positions and differing from one another in the i^{th}. From Prop 2, block i has influence on both or neither of v_+ and v_-. The condition for it to have influence on both is $|\sum_j n_j \cdot v_j| \leq n_i$ and is satisfied by assumption. The contribution to the sum in equation 2 is therefore $P(v_+) + P(v_-)$. From its definition, which explicitly incorporated such conditioning, $P_{-i}(v') = P(v_+) + P(v_-)$. Our two terms yield the same contribution as the single term in the equation 1. □

In calculation, it is immaterial whether we use the full or reduced space and distribution. However, we will discuss their respective interpretations in section 4.4.

4.2. **Properties of Influence.** The Influence of a block I_i depends on the structure of the system B only through the set of elections on which i has influence. Any invariance of that set is an invariance of I_i too. The following are some general properties of I:

Proposition 14. *:*

1. $I_i = 0$ *if* i *is irrelevant*[23].
2. $I_i = 1$ *for a participant with* $n_i > \sum_{j \neq i} n_j$ *and* 0 *for all others.*
3. I *depends only on ratios of allotments and is invariant if we scale all* n_i *by a constant factor.*
4. I *depends on* P *only through the symmetric form* $P(v) + P(-v)$.

Proof. :

1. There is no election on which i has influence, so I_i is a sum over the empty set and is 0.
2. From Prop 4, the largest block would be relevant and all others irrelevant. However, the latter means that n_1 always decides the election and has influence on all elections. Thus, $I_1 = \sum_{v \in S_B} P(v) = 1$. The others are irrelevant and, as previously shown, have $I_i = 0$.
3. As in the proof of Prop 4, the condition $|\sum_{j \neq i}| \leq n_i$ is invariant under scaling of n. The set of elections on which i has influence is unchanged. This is the only structural dependency of I, as the summand $P(v)$ is independent of n.
4. From Prop 2, we know that i has influence on v iff it has influence on $-v$. Thus, I_i can be written as a sum over the elections on which i has influence and $v_j > 0$ for some j, with $P(v) + P(-v)$ in the summand.

□

Note that blocks of the same size are not necessarily fungible.

4.3. **Choice of Measure.** Any calculation of Influence requires a choice of measure P over S_B. There are some constraints on P.

1. $P = \prod_i P_i$. We assumed voters act independently, and this requires that the probability of a given election be a product of independent probabilities.
2. We assumed a binary election, so each P_i is a Bernoulli distribution. We denote by p_i the probability of participant i voting $+1$.

We may write $P(v) = \prod_{i|v_i=-1} (1 - p_i) \cdot \prod_{i|v_i=+1} (p_i)$ which can be expressed more succinctly as $P(v) = \prod_i (p_i v_i + \frac{1}{2}(1 - v_i))$. For convenience, we denote the argument in the product as $q_i(v_i) \equiv (p_i v_i + \frac{1}{2}(1 - v_i))$. This may also be written $q_i(v_i) = (\frac{1}{2} + (p_i - \frac{1}{2})v_i)$.

As expected, identical blocks are fungible if their probabilities are equal.

Proposition 15. *If* $n_i = n_j$ *and* $p_i = p_j$ *then* $I_i = I_j$.

Proof. Suppose $n_i = n_j$ and $p_i = p_j$. Let n_j have influence on v. Then v appears in the domain over which we sum equation 1 for I_j. Define v' to equal v but with v'_i and v'_j swapped. Then, $|v_i \cdot n_i + \sum_{k \neq i,j} n_k \cdot v_k| \leq n_i$ implies that $|v'_j \cdot n_j + \sum_{k \neq i,j} n_k \cdot v'_k| \leq n_j$. There is a unique v' on which n_i has influence for every v on which n_j has influence.

[23] The converse, that it is 0 only in this case, holds as long as the support for P is the entirety of S_B. In section 4.3 it will be clear that this is true iff each individual block has probability $0 < p_i < 1$ of voting $+1$. In the nomenclature we will introduce, this means there are no fixed blocks.

Moreover, $q_i(v_i) = q_j(v'_j)$ and therefore $P(v) = q_i(v_i) \cdot \prod_{k \neq i,j} q_k(v_k) = P(v')$. As $n_j = n_i$, the converse holds too. There is a one-to-one correspondence between terms in the sums for I_i and I_j, so the two must be equal. □

4.4. **Interpretation of p_i.** We specify p_i as the probability that participant i will vote +1 for the purpose of determining the Influence of others. It does not appear in the computation of block i's Influence. The definition of Influence presumes that each voter could vote either way. This comes back to section 1.6 and our choice of admissible actions. We are concerned with ex-ante rather than ex-post probabilities. In computing I_i, we are not estimating the probability that i could have changed the outcome of an election. We are estimating the probability that the confluence of other voters' choices places him in a position to decide the outcome of a future election. His choice will be made independently, without regard for those votes. As shown by Prop 13, either approach yields the same result.

An interesting special case is that of fixed blocks, discussed in section 4.8, which have $p_i = 0$ or $p_i = 1$. In that case, one may ask whether it makes sense to assume such predetermination in the calculation of other Influences, but allow the choice implicit in the calculation of his own. It does not. Though calculable, the Influence of such a block is not a useful quantity to consider. As will be discussed, the presence of such blocks really should be treated structurally rather than through a choice of probability distribution.

The need to consider reduced sample spaces can lead to some surprising results. Consider a system with 3 blocks of sizes $(4, 3, 2)$ and respective probabilities $(0.2, 0.8, 0.8)$. In calculating the Influence of the 2^{nd} block, there are two reduced elections on which it has influence: $(+1, \times, -1)$ and $(-1, \times, +1)$ with probabilities of 0.8×0.8 and 0.2×0.2. The Influence is $I_2 = 0.64 + 0.04 = 0.68$. Similarly, there are two reduced elections on which the 1^{st} block has influence: $(\times, +1, -1)$ and $(\times, -1, +1)$ with probabilities 0.2×0.8 each. The Influence is $I_1 = 0.16 + 0.16 = 0.32$. The largest block has far less Influence than the next largest. Because we are sampling over a different probability distribution $(p_1 \cdots p_{i-1}, p_{i+1}, \cdots p_N)$ in the calculation of each Influence I_i, it is possible for such anomalies to arise. This is not a theoretical eccentricity; we will encounter it in analyzing the US electorate in section 5.5.

4.5. **The Case of Equal Probabilities $p_i = p$.** If we assume that each participant's probability of voting +1 is the same (though not necessarily 0.5), then certain convenient properties hold. To compute Influence I_i we sum over the set of reduced elections $(v_1 \cdots v_{i-1}, v_{i+1} \cdots v_N)$ such that $|\sum_{j \neq i} n_j \cdot v_j| \leq n_i$. Denoting by $N_-(v')$ the number of votes that are -1 in a given election v', we have $P(v') = p^{N-1}(\frac{1-p}{p})^{N_-(v')}$. The v' we consider and their associated probabilities are the same for all i (with a formal change of indices)[24]. Denoting by $x(k, m)$ the number of elections v' such that $|\sum_k n_k \cdot v_k| \leq m$ and $N_-(v') = k$, we can write the

[24]This is only true in the special case of equal probabilities. In fact, the anomalies mentioned earlier arise precisely because this does not hold for general p_i.

Influence as

$$I_i = p^{N-1} \cdot \sum_{j=0}^{N} x(j, n_i) \cdot \left(\frac{1-p}{p}\right)^j \tag{3}$$

Proposition 16. *For equal probabilities $p_i = p$,*

1. *Identical blocks are fungible. If $n_i = n_j$, then $I_i = I_j$.*
2. *Influence is a non-decreasing function of block size. If $n_i > n_j$, then $I_i \geq I_j$.*

Proof. :

1. This is a special case of Prop 15.
2. Suppose $n_i > n_j$. Let n_j have influence on election v. Then $-n_j \leq v_i \cdot n_i + \sum_{k \neq i,j} n_k \cdot v_k \leq n_j$. Prop 2 assures us that n_j has influence on $-v$, so we may choose whichever of v and $-v$ has $v_i > 0$ without loss of generality. From $-n_j \leq n_i + \sum_{k \neq i,j} n_k \cdot v_k$ we readily obtain $-n_i \leq n_j + \sum_{k \neq i,j} n_k \cdot v_k$. We also observe $n_j + \sum_{k \neq i,j} n_k \cdot v_k \leq n_i + \sum_{k \neq i,j} n_k \cdot v_k \leq n_j < n_i$. Thus, $-n_i \leq n_j + \sum_{k \neq i,j} n_k \cdot v_k \leq n_i$ and n_i has influence on v', defined as v but with $v'_i = v_j$ and $v'_j = v_i$. Moreover, $N_-(v)$ is unchanged and the probability of that election is the same in the calculation of both I_i and I_j. The domain of the sum for I_j is a subset of that for I_i, and the summand is the same on it and non-negative everywhere else. Therefore $I_i \geq I_j$.

□

4.6. **The Case of 3 Blocks.** To illustrate the general effect of probabilities, consider the simplest systems. We have 3 blocks with sizes (n_1, n_2, n_3) and probabilities (p_1, p_2, p_3). If $n_1 > n_2 + n_3$ we have trivial dominance by one participant, so assume that $n_1 \leq n_2 + n_3$. Each participant has influence on any reduced election in which the other voters oppose one another. For example, $I_1 = p_2(1 - p_3) + p_3(1 - p_2)$. The Influences are:

$$I_1 = p_2 + p_3 - 2p_2p_3$$
$$I_2 = p_1 + p_3 - 2p_1p_3$$
$$I_3 = p_1 + p_2 - 2p_1p_2$$

If $p_1 = p_2 = p_3 = p$, then $I_1 = I_2 = I_3 = 2p(1 - p)$. For $p = 0.5$, this yields $I_i = 0.5$.

For higher N, the combinatorics grows considerably worse, and analytical expressions are unenlightening.

4.7. **Effect of Block Modification.** We can say a few things about the Influence of blocks when merged, split, or changed in size. Note that we must specify the change in P as well as S_B.

Proposition 17. *For a system B and probability distribution P, consider the effect of the following operations. Denote the resulting system B' and its associated Influences I'.*

1. *If $n_i \to n_i + 1$ and p_i is unchanged, then $I'_i \geq I_i$.*
2. *If $n_i \to n_i - 1$ and p_i is unchanged, then $I'_i \leq I_i$.*

(3) *If two blocks n_i and n_j are merged into a block n_c (with any choice of probability p_c), then $I'_c \geq \max(I_i, I_j)$*
(4) *If a block n_c is split into two blocks n_i and n_j with associated probabilities p_i and p_j, then in the combined system:*
 (a) $I_c \cdot \min(p_j, 1-p_j) \leq I'_i \leq I_c$
 (b) $I_c \cdot \min(p_i, 1-p_i) \leq I'_j \leq I_c$

Proof.
(1) If n_i has influence on v, then $|\sum_{j\neq i} n_j \cdot v_j| \leq n_i \leq n_i + 1$, so $n_i + 1$ has influence too. The domain of the sum is at least as large, with a non-negative summand that is unchanged, so $I'_i \geq I_i$.
(2) This follows from the previous result. If $I'_i > I_i$, then adding 1 back to $n_i - 1$ would result in $I''_i \geq I' > I_i$, but $I''_i = I_i$ which contradicts this. So, $I'_i \leq I_i$.
(3) Consider an election v on which n_i has influence in B. Let $S(v) = \sum_{k\neq i,j} n_k \cdot v_k$ and let $Q(v) = \prod_{k\neq i,j} q_k(v_k)$. Then, $-n_i \leq S(v) + n_j \cdot v_j \leq n_i$. For both $v_j = \pm 1$, we have $-n_i - n_j \leq S(v) \leq n_i + n_j$, so the combined block has influence on the reduced election v' defined as v excluding v_i and v_j. It is possible that this is also the case for the same v but with v_j flipped - meaning that 2 elections in B could map to the same one in B'. The summand in the merged system is simply $Q(v)$. In the original system, the two summands are $p_j \cdot Q(v)$ and $(1-p_j) \cdot Q(v)$. All combinations of one or more of these are $\leq Q(v)$. The merged sum is over at least the same domain and has at least as large a summand where they overlap (and positive summand everywhere else). Therefore $I'_c \geq I_i$. The same holds for I_j.
(4) That both I'_i and I'_j are $\leq I_c$ follows from the previous result; just recombine n_i and n_j into n_c. As for the other limit, let us revisit our previous analysis. If n_c has influence on v', define $Q(v')$ and $S(v')$ the same way as before but excluding only n_c. I.e. our sum is over v' such that $|S(v')| \leq n_c$. It is not hard to see that n_i has influence on either one or both of the reduced elections $(v', v_j = \pm 1)$. For example, if $0 \leq S(v) \leq n_c = n_i + n_j$ then $|S(v) - n_j| \leq n_i$. It is possible but not certain that $|S(v) + n_j| \leq n_i$ as well. The opposite is true if $-n_c \leq S(v) \leq 0$. The two possible summands are $p_j Q(v)$ and $(1-p_j)Q(v)$. The smallest possible value is $\min(p_j, 1-p_j)Q(v)$. I'_i is therefore a sum of terms that are greater than or equal to $\min(p_j, 1-p_j)$ times the associated terms in I_c. So $I'_i \geq \min(p_j, 1-p_j)I_c$. The same holds for I'_j.

□

Note that this doesn't mean that we can compare Influences in an existing system. For example, in our Electoral example below, if we look at the three blocks of size $(55, 34, 21)$ we can't apply the theorem. If we actually merged the 34 and 21 blocks into a new block of size 55, the theorem would hold regarding the old Influences of the 34 and 21 blocks and the new 55 block - with no reference to the other 55 block that was unchanged between the systems.

4.8. **Effect of Fixed Blocks.** If certain blocks always vote one way, the problem is analogous to one in which $O(v)$ is determined by the sign of $X(v)-T$ for some T. We incorporate such "fixed blocks" into our analysis by assigning them probability 1 or 0. Conversely, we may analyze a broader class of problems within our framework. Suppose a system has elections conforming to our definition but the outcome is determined by other than a simple plurality. We can recast it as a system decided by simple majority but with one extra block. For example, if the 100 members of the US Senate require a 2/3 majority to override a veto, then $T = 33$. The analogous system has 101 participants, of which 100 have 1 vote each and the other has 33 votes and chooses -1 in each election with probability 1.

As mentioned, the Influence of a fixed block itself is not a useful quantity. The following proposition justifies their aggregation when computing the Influence of other blocks.

Proposition 18. *Let the respective sums of the fixed blocks that vote* $+1$ *and* -1 *be* n_+ *and* n_-*. In any systems with the same set of non-fixed blocks (along with associated probabilities) and the same value of* $n_+ - n_-$*, the Influences of non-fixed blocks will be the same.*

Proof. Consider a non-fixed block n_i. Its Influence is given by equation 1. The summand is $P(v) = \prod_j q_j(v_j)$. It is only nonzero when all the fixed blocks vote in their prescribed directions. Fixed blocks have $v_j = +1$ and $p_j = 1$ or $v_j = -1$ and $p_j = 0$, so $q_j(v_j) = 1$ in both cases. Denote the set of non-fixed blocks Q. The summand, where nonzero, is $P(v) = \prod_{j \in Q} q_j(v_j)$ and is independent of the number or sizes of fixed blocks. The domain is $|\sum_{j \neq i} n_j \cdot v_j| \leq n_i$, which for nonzero $P(v)$ is $|n_+ - n_- + \sum_{j \neq i \in Q} n_j \cdot v_j| \leq n_i$. The only dependence on the fixed blocks is through the difference of their sums. □

We may merge, split, expand, shrink, add, and subtract fixed blocks. As long as $n_+ - n_-$ is unchanged, the Influences of non-fixed blocks will be unaffected.

4.9. **A Model for Abstention.** We can use our framework to map certain voting systems with abstention. Consider a system with two parties A and B. Suppose that individuals (or blocks) belong to one of the two parties and always vote for their party if they vote at all. The choice each individual makes is whether or not to vote. We map the problem as follows: an individual of party A is deemed to have voted $+1$ if they show up and -1 if not, while the opposite is true for an individual of party B. The choice is still a binary one and the party affiliations are fixed, so there is no problem with assigning differing meanings to the votes. Let the sum of all blocks that show up for party A be denoted n_a and that for party B be n_b. For any election v, the outcome is determined by $X(v) = n_a - n_b$. The problems are therefore isomorphic.

5. The Electoral College

As an illustrative example of an Influence calculation, we examine the case of the US electoral college. In a US Presidential election, voting takes place in two stages. A popular vote within each state determines the choice of electors for that state. The electoral college, consisting of all the states' chosen electors, then convenes and votes separately for the President and the Vice President. The number of electors allocated to each state is the sum of the number of senators (2) and the number of representatives in the House (determined by the most recent census)[25] (see [5]). In theory, the details of the popular votes are determined by the states themselves[26]. In practice the popular ballot lists candidates for President, and the vote for the corresponding parties' electors is implicit[27].

5.1. **Data.** Data for the historical distribution of electoral votes was obtained from the Federal Register [2]. We made a few simplifying assumptions to fit the real electoral process into our framework.

(1) We assumed a two-party system. Additional candidates were ignored.
(2) We assumed all states cast votes in indivisible blocks.
(3) We assumed electoral delegates are obligated to vote as their party chooses.
(4) In the 1988 election, 1 delegate from West Virginia voted for Lloyd Bentsen and the rest voted for Dukakis. We pushed that vote to Dukakis.
(5) In the 2000 election, Washington DC cast only 2 out of 3 votes. There was one abstention. We treated them as having cast all 3 votes in the same direction.
(6) In the 2004 election, 1 delegate from Minnesota voted for John Edwards for President and the rest voted for John Kerry. We pushed that vote to Kerry.

Note that the 2008 electoral allocation is the same as that of 2004. Both are based on the 2000 census.

5.2. **Probabilities.** We examined Influences of the states in the current (2008) electoral college using both even probabilities and historically derived probabilities. The latter were calculated as the fraction of the last 6 elections (ending in 2004) that each state voted for a republican[28]. It may be tempting to aggregate fixed blocks (those that voted the same way in all 6 elections) into two party blocks and examine the problem in that context. However, as observed in Prop 18 this adds no new information.

5.3. **Sampling.** We employ Monte-Carlo sampling using a pseudo-random generator based on the GNU Scientific Library (MT19937 algorithm, see [3]). A fixed seed of our choosing was employed for reproducibility. Our base sample size was 10^6 elections[29]. There are a few issues that arise in such a sampling:

(1) There are two ways to sample elections for a given block and set of participant probabilities p_i. (i) We may randomly select an election using 0.5 probability for each participant's vote and then weigh the election by the

[25] The District of Columbia has the same number of electors as the smallest state.

[26] Almost all states allocate choose their entire set of electors based on a simple popular majority. Maine and Nebraska assign the 2 "senatorial" electors that way and determine the remaining "representative" electors by the popular outcome in each district. Theoretically, their blocks could be split.

[27] Though the vote by the electoral college itself is largely formal, it is sometimes possible for an elector to vote differently than charged by his party.

[28] Though it might be of interest to perform the same analysis using the controversial last 2 elections, most states would have 0 or 1 probabilities and the influence statistics would not be informative.

[29] We sample with replacement.

appropriate overall probability in our calculations or (*ii*) We may randomly select an election using the given probabilities and assign it equal weight in our calculations. For a large enough sample, the results will be the same; however the latter concentrates our samples where most of the probability resides, and is computationally more efficient[30].

(2) Regardless of how we draw our sample, we select a full election vector. Then, for each participant we treat the associated reduced vector (excluding its entry) as an independent sample. This is because the Influence of participant i is really obtained by sampling over the possible decisions of the other $N-1$ participants. While we could ignore this and get the same answer for large enough samples, that would be wasteful. We weigh each election by $\frac{P}{q_i}$ (where $q_i = p_i$ if $+1$ and $q_i = (1-p_i)$ if -1) to remove the probability assigned to vote v_i from the calculation of its own Influence.
(3) We expect the Influences of identical blocks to differ. This results partly from differences in their voting probabilities, but also from the random sampling of elections. As we cannot separate the two, statistical differences between block Influences may arise even in cases where their probabilities are the same.

5.4. **Electoral Influence.** Table 12 provides the Influence vectors for the current US electorate using both of our assumed probability distributions[31]. The columns are:

- "Inf:equal": The Influence vector assuming that each state is equally likely to vote for either party.
- "Inf:prob": The Influence vector using the fraction of the past 6 elections in which each state voted for a Republican as the probability for that state.

Figures 1 and 2 display the Influence vector as well as the associated Influence per electoral vote as a function of block size in each case. A linear best fit is included as well. We do not expect the curve to be linear, but this provides a handy reference level.

The stability of the results is excellent. To understand convergence of the simulation, we performed the Monte Carlo analysis using a sequence of six samples, differing by powers of 2 in size[32]. This gave rise to a sequence of Influence vectors which could then be compared. The variation was minimal between the base ($k = 10^6$) and the smallest sample ($k = 31,250$). We also measured the maximum absolute difference between any component of the base sample and the corresponding component in any other sample. This turned out to be around 0.006 for the equal probability case and 0.007 for the historical one, each of which is $< 25\%$ of the respective smallest component's Influence.

5.5. **Analysis of Results.** As long as all blocks are significantly smaller than the total, we intuitively expect that Influence should increase linearly with size in the

[30]The statistics are well behaved for a product of Bernoulli distributions. Were we sampling an arbitrary distribution, such a strategy could prove deceptive. However, this doesn't pose a problem in our analysis for any reasonable sample size.

[31]We also include the probabilities themselves as well as a "Party" column which indicates whether a state voted the same in the last 6 elections.

[32]We did this separately for equal probabilities and historical probabilities.

Table 12. Influence Statistics for Electoral Votes (2004 Electorate)

State	Votes	$P(+1)$	Party	Inf:equal	Inf:prob
California	55	0.33333	-	0.482760	0.235185
Texas	34	1.00000	R	0.274370	0.313074
New York	31	0.16667	-	0.249404	0.158020
Florida	27	0.83333	-	0.216911	0.241949
Illinois	21	0.33333	-	0.169136	0.139955
Pennsylvania	21	0.33333	-	0.169061	0.139986
Ohio	20	0.66667	-	0.161299	0.163547
Michigan	17	0.33333	-	0.137599	0.117091
New Jersey	15	0.33333	-	0.122320	0.105182
North Carolina	15	1.00000	R	0.122513	0.131039
Georgia	15	0.83333	-	0.122349	0.126039
Virginia	13	1.00000	R	0.106669	0.112885
Massachusetts	12	0.16667	-	0.098704	0.082160
Washington	11	0.16667	-	0.091543	0.075670
Missouri	11	0.66667	-	0.091269	0.087993
Tennessee	11	0.66667	-	0.091421	0.088435
Indiana	11	1.00000	R	0.091595	0.095202
Maryland	10	0.33333	-	0.083780	0.074174
Arizona	10	0.83333	-	0.083957	0.083401
Minnesota	10	0.00000	D	0.083771	0.066600
Wisconsin	10	0.16667	-	0.083482	0.070235
Colorado	9	0.83333	-	0.076128	0.074985
Louisiana	9	0.66667	-	0.076352	0.073001
Alabama	9	1.00000	R	0.076259	0.077899
Kentucky	8	0.66667	-	0.068907	0.064764
South Carolina	8	1.00000	R	0.068709	0.069486
Oregon	7	0.16667	-	0.061027	0.052455
Connecticut	7	0.33333	-	0.061104	0.054594
Oklahoma	7	1.00000	R	0.060977	0.061376
Iowa	7	0.33333	-	0.060580	0.054157
Mississippi	6	1.00000	R	0.053257	0.053146
Kansas	6	1.00000	R	0.053341	0.053146
Arkansas	6	0.66667	-	0.053290	0.050317
Nebraska	5	1.00000	R	0.045544	0.045095
West Virginia	5	0.50000	-	0.045923	0.042018
Nevada	5	0.66667	-	0.045681	0.043052
New Mexico	5	0.50000	-	0.045548	0.042130
Utah	5	1.00000	R	0.045812	0.045095
Hawaii	4	0.16667	-	0.038028	0.033960
Maine	4	0.33333	-	0.038226	0.034270
New Hampshire	4	0.50000	-	0.038300	0.035032
Rhode Island	4	0.16667	-	0.038187	0.033753
Idaho	4	1.00000	R	0.038092	0.037190
North Dakota	3	1.00000	R	0.030627	0.029420
Delaware	3	0.33333	-	0.030450	0.027688
District of Columbia	3	0.00000	D	0.030789	0.026600
Vermont	3	0.33333	-	0.030759	0.027663
South Dakota	3	1.00000	R	0.030627	0.029420
Montana	3	0.83333	-	0.030638	0.028747
Alaska	3	1.00000	R	0.030459	0.029420
Wyoming	3	1.00000	R	0.030412	0.029420

even probability case[33]. This is almost the case. However, as shown in Figure 1 the very small states have higher influence-per-vote than the rest[34].

On the other hand, it is clear from Figure 2 that Influences behave nonlinearly for the larger blocks when we employ historically derived probabilities. They do not even increase monotonically. In particular, CA and NY appear to have less Influence per vote than the mean and TX and FL appear to have more. This is not as counterintuitive as it may seem. As mentioned in section 4.4, such behavior

[33]Note that states with the same probabilities and numbers of votes have Influences that differ infinitesimally, if at all. This is a good sign, but less so than might be expected. Because we used the same elections for sampling the influence of all states, elections on which a small state has influence will have outcome close to 0. Therefore, it is very likely that other small states will also have influence. The results are accurate, but the probability of coincident influences of small states is high due to our method.

[34]This would appear to accentuate the discrepancy in per-person influence between small and large states, a source of controversy.

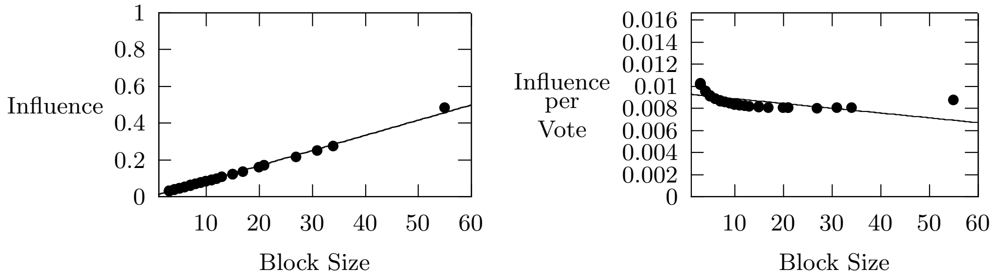

FIGURE 1. Even Probabilities

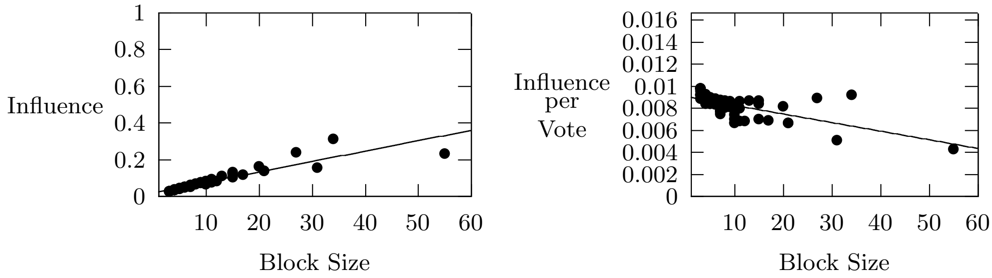

FIGURE 2. Historic Probabilities

arises from the differing reduced probability distributions used in the consideration of blocks' Influences. In the case of the electoral college, large states that favored the historically less successful party (the Democrats) exert less Influence under our probability distribution.

6. Generalization

We may extend our definitions to a wide variety of voting systems[35]. To do so, we require the following concepts:

(1) A voting system B, with a set of N defined participants who make independent decisions without the potential for gaming[36].
(2) An election v as a set of decisions for all participants.
(3) A set of outcomes $\{O(v)\}$ for the election.
(4) A sample space S_B of admissible elections.

Example. *For example, in one fictitious system each of* 53 *participants may have a list of* 7 *candidates. Their choice consists of assigning each candidate a weight, with the total equal to the participant's age. An election consists of such a decision for each of the* 53 *participants. The outcome could be determined by summing the*

[35]Though the meaning is unambiguous, the actual mathematical formulations will depend on the specifics of each given system.

[36]This may involve the imposition of blind, simultaneous voting.

weights assigned to each candidate and selecting the highest scoring as President and the lowest as VP.

Our definitions are

(1) Participant i **has influence on election** $v \in S_B$ if any change in his choice, resulting in an election $v' \in S_B$, would have $O(v') \neq O(v)$.
(2) Participant i is **relevant** if there exists an election $v \in S_B$ on which he has influence.
(3) Participant i is **irrelevant** if he is not relevant.
(4) A **relevance vector** r can be associated with B. It is 1 if participant i is relevant and 0 otherwise.
(5) Given a probability measure P over S_B, the **Influence of participant** i **with respect to** P is the probability that i has influence on an election sampled from S_B using P. Clearly P must still be a product of independent distributions, but these need not be Bernoulli.
(6) The **Influence Vector** $I(B, P)$ has the Influence of participant i with respect to P as its i^{th} element.

Example. *In our example, an election* $v \in S_B$ *consists of* 53 *sets of* 7 *numbers, each of which sums to the age of the corresponding participant. Suppose that all candidates are the same age (*100 *for simplicity) and that the sums over* v *are :* $(2037, 1976, 699, 307, 153, 116, 12)$. *No participant can influence the VP slot because that would require* 104 *votes. However, a participant who voted* 80 *for the* 3^{rd} *place candidate could change the choice of President by reallocating all votes to the* 2^{nd} *place candidate. Likewise a candidate who attached* 37 *votes to the* 1^{st} *place candidate and* 63 *to the* 2^{nd} *could have changed the outcome by assigning all her votes to the* 2^{nd} *place one. Both such participants (had they voted as specified) would have influence on* v. *Both are relevant. In fact, in such a system there are no irrelevant participants*[37].

It is clear that our definition of Influence is a general one.

References

[1] I.S. Gradshteyn and I.M. Ryzhik *Tables of Integrals, Series, and Products, Corrected and Enlarged Edition*, San Diego, Academic Press, 1980, Section 0.151.1.
[2] Federal Register Historical Election Results `http://www.archives.gov/federal-register/electoral-college/votes/votes_by_state.html`
[3] GNU Scientific Library `http://www.gnu.org/software/gsl/manual/html_node/index.html`
[4] Milton Abramowitz and Irene A. Stegun *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, New York, National Bureau of Standards, 1964 (10th Printing, 1972), Section 24.2.1 and Table 24.5.
[5] *Constitution of the Unites States*, Article II, Section 1, the 12^{th} Amendment, and the 23^{rd} Amendment.

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[37] An election exists with the last place candidate receiving 1 vote. Every participant has influence on such an election.